BEST LINEAR UNBIASED ESTIMATION AND PREDICTION OF ORDER STATISTICS AND RECORD STATISTICS

DISSERTATION

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Dedicated To

My Parents

&

My Supervisor

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(Salma Iqrar)

CONTENTS

PREFACE

The present dissertation entitled **"Best Linear Unbiased Estimation and Prediction of Ordered Statistics and Record Statistics"** is a brief collection of the work done so far on the subject. I have tried my best to include sufficient and relevant materials in the systematic way, which are contained in five chapters.

Chapters I is introductory in nature in which some concepts which may be needed to grasp the ideas contained in the remaining chapters are discussed.

Chapter II consists of some recurrence relations for single and product moments for some specific continuous truncated distributions for order statistics and record values. *viz.* Weibull, exponential, Pareto, generalized Pareto, power function, and uniform distributions.

In chapter III, BLUEs of scale and location parameters for exponential, generalized Pareto, logistic and uniform distributions have been considered using order statistics.

Chapter IV includes BLUEs of scale and location parameters for exponential, generalized Pareto, power function, Weibull and Rayleigh distributions based on record values.

In Chapter V, BLUPs of scale and location parameters for exponential, generalized Pareto, power function, Weibull and Rayleigh distribution for order statistics and record values have been embodied.

In the end, a comprehensive list of references referred into this dissertation is given.

CHAPTER I

PRELIMINARIES AND BASIC CONCEPTS

In this chapter we have included those concepts and results, which are needed to grasp the contents in subsequent chapters.

1. ORDER STATISTICS

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. Let they be arranged in ascending order of magnitude as

$$X_{1:n} \leq X_{2:n} \leq ..... \leq X_{r:n} \leq ..... \leq X_{n:n}$$

then $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are collectively called the order statistics of the sample and $X_{r:n}$ $(r = 1,2,...,n)$ is called the r^{th} order statistic of the sample. $X_{1:n} = \min(X_1, X_2, ..., X_n)$ and $X_{n:n} = \max(X_1, X_2, ..., X_n)$ are called extreme order statistics or the smallest and the largest order statistics. r^{th} order statistics is also denoted by $X_{(r)}$, if the size of the sample is fixed.

1.1: PROBABILITY DENSITY FUNCTION (pdf) OF A SINGLE ORDER STATISTIC

The pdf of $X_{r:n}$, the r^{th} order statistic is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \,,$$
$$-\infty < x < \infty \,. \qquad (1.1)$$

The $pdf's$ of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x) \;;\; -\infty < x < \infty \qquad (1.2)$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) \;;\; -\infty < x < \infty \qquad (1.3)$$

1.2: CUMULATIVE DISTRIBUTION FUNCTION (*df*) OF A SINGLE ORDER STATISTIC

The df of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \le x)$$

$= P$ (at least r of $X_1, X_2, ..., X_n$ are less than or equal to x)

$= \sum_{i=r}^{n} P$(exactly i of $X_1, X_2, ..., X_n$ are less than or equal to x)

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} \;;\; -\infty < x < \infty \qquad (1.4)$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du \qquad (1.5)$$

$$= \mathrm{I}_{F(x)}(r, n-r+1) \qquad (1.6)$$

where $I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt, \;\; B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$

RHS in (1.5) is obtained by the relationship between binomial sums and incomplete beta function. It may also be expressed in negative binomial sums as (Khan, 1991)

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{i+r-1}{r-1} \{F(x)\}^r \{1-F(x)\}^i, \quad -\infty < x < \infty \qquad (1.7)$$

For continuous case the *pdf* of $X_{r:n}$ may also be obtained by differentiating (1.5) *w.r.t.* x.

The k^{th} moment of $X_{r:n}$ is

$$\alpha_{r:n}^{(k)} = E[X_{r:n}^k] = \int_{-\infty}^{\infty} x^k f_{r:n}(x)dx \tag{1.8}$$

1.3: JOINT PROBABILITY DENSITY FUNCTION OF TWO ORDER STATISTICS

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\times[1-F(y)]^{n-s} f(x)f(y)\,; \quad -\infty < x < y < \infty \tag{1.9}$$

Thus the joint *pdf* of 1^{st} and n^{th} order statistics is

$$f_{1,n:n}(x,y) = \frac{n!}{(n-2)!}[F(y)-F(x)]^{n-2} f(x)f(y).$$

1.4: JOINT CUMULATIVE DISTRIBUTION FUNCTION OF TWO ORDER STATISTICS

The joint *df* of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ can be obtained as follows:

$$F_{r,s:n}(x,y) = P(X_{r:n} \le x, X_{s:n} \le y)$$

$= P$ (at least r of $X_1, X_2, ..., X_n$ are at most x

and at least s of $X_1, X_2, ..., X_n$ are at most y)

$= \sum_{j=s}^{n} \sum_{i=r}^{j} P$(exactly i of $X_1, X_2, ..., X_n$ are at most x

and exactly j of $X_1, X_2, ..., X_n$ are at most y)

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!}[F(x)]^i$$

$$\times [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j} \tag{1.10}$$

We can write the joint *df* of $X_{r:n}$ and $X_{s:n}$ in (1.10) equivalently as:

$$F_{r,s:n}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{F(x)} \int_0^{F(y)} u^{r-1} (v-u)^{s-r-1}$$

$$\times (1-v)^{n-s} du dv$$

$$= I_{F(x),F(y)}(r, s-r, n-s+1) \, ; \, -\infty < x < y < \infty \tag{1.11}$$

which is incomplete bivariate beta function.

1.5: Joint Probability Density Function of More Than Two Order Statistics

The joint *pdf* of k order statistics $X_{r_1}, X_{r_2}, \ldots, X_{r_k}$, where $1 \leq r_1 \leq r_2 \leq \ldots \leq r_k \leq n$ and $1 \leq k \leq n$,is given by

$$f_{r_1, r_2, \ldots, r_k}(x_1, x_2, \ldots, x_k)$$

$$= \frac{n!}{(r_1 - 1)!(r_2 - r_1 - 1)! \ldots (r_k - r_{k-1} - 1)!(n - r_k)!}$$

$$\times [F(x_1)]^{r_1 - 1} f(x_1) [F(x_2) - F(x_1)]^{r_2 - r_1 - 1} f(x_2) \cdots$$

$$\times [F(x_k - F(x_{k-1})]^{r_k - r_{k-1} - 1} f(x_k) [1 - F(x_k)]^{n - r_k},$$

$$x_1 \leq x_2 \leq \ldots \leq x_k, \tag{1.12}$$

In particular, the joint *pdf* of all the n order statistics is obtained on taking $k = n$ in (1.12).

Hence the joint *pdf* of $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ is given by

$$f_{1,2,\ldots,n:n}(x_1, x_2, \ldots, x_n) = n! f(x_1) f(x_2) \cdots f(x_n). \tag{1.13}$$

1.6: CONDITIONAL DISTRIBUTION OF ORDER STATISTICS

The conditional *pdf* of $X_{s:n}$ given $X_{r:n} = x$, ($r < s$) is

$$\frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)}{[1-F(x)]^{n-r}}$$

$$x \le y$$

which is just the unconditional *pdf* of the $(s-r)^{th}$ order statistic in a sample of size $(n-r)$ drawn from $\frac{f(y)}{1-F(x)}$, $y \ge x$, that is from the parent distribution truncated on the left at x. Also, the conditional distribution of $X_{r:n}$ given $X_{s:n} = y$ ($r<s$) is

$$\frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{r-1} f(y)}{\{F(x)\}^{s-1}}, x \le y$$

which is just the unconditional *pdf* of the r^{th} order statistic in a sample of size $(s-1)$ truncated on the right at y.

2. RECORD VALUES AND RECORD TIMES

Suppose that $X_1, X_2, \cdots, X_n$ is a sequence of independent and identically distributed random variables with *df* $F(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}$ for $n \ge 1$. We say X_j is an upper (lower) record values of $\{X_n, n \ge 1\}$, if $Y_j > (<) Y_{j-1}, j > 1$. By definition X_1 is an upper as well as lower record values. One can transform the upper record by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \ge 1\}$ or if $\Pr(X_i > 0) = 1$ for all i by $\left\{\frac{1}{X_i}, i \ge 1\right\}$, the lower record value of this sequence will correspond to the upper record values of the original sequence (Ahsanullah, 2004).

The indices at which upper record values occur are given by the record times $\{U_{(n)}\}, n > 0$. That is $X_{U(n)}$ is the n^{th} upper record, where $U_{(n)} = \min\{j \mid j > U_{(n-1)}, X_j > X_{U(n-1)}, n > 1\}$ and $U_{(1)} = 1$. The distribution of $U_{(n)}, n \geq 1$ does not depend on F. Further, we will denote $L_{(n)}$ as the indices where the lower record values occur. By assumption, $U_{(1)} = L_{(1)} = 1$. The distribution of $L_{(n)}$ also does not depend on F.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: e.g. Olympic records or world records in sports.

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.

To be precise, record values are defined by means of record times. That is, those times have to be described at which successively largest values appear.

Chandler (1952) has given several properties of record values including their Markovian structure. Two recent books on records by Ahsanullah (2004) and Arnold *et al.* (1998) are worth mentioning.

2.1: DISTRIBUTION OF RECORD VALUES

Let $R(x)$ be a continuous function of x with $R(x) = -\ln \bar{F}(x)$ and $0 < \bar{F}(x) = 1 - F(x)$, where 'ln' is the natural logarithm.

If we define $F_n(x)$ as the *df* of $X_{U(n)}$ for $n \geq 1$, then we have (Ahsanullah, 1995)

$$F_n(x) = P(X_{U(n)} \leq x) = \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \; -\infty < x < \infty \tag{2.1}$$

and the *pdf* $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), -\infty < x < \infty. \tag{2.2}$$

The joint *pdf* of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i,j}(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j),$$

$$-\infty < x_i < x_j < \infty. \tag{2.3}$$

The joint *pdf* of the n-record values $X_{U(1)}, X_{U(2)}, \cdots, X_{U(n)}$ is given by

$$f_{1,2,\cdots,n}(x_1, x_2, \ldots, x_n) = r(x_1)\, r(x_2) \cdots r(x_{n-1}) f(x_n),$$

$$-\infty < x_1 < x_2 < \cdots < x_{n-1} < x_n < \infty, \tag{2.4}$$

where $r(x) = \dfrac{dR(x)}{dx} = \dfrac{f(x)}{\bar{F}(x)}$, $0 < F(x) < 1$ is known as hazard rate.

In particular at $i = 1$, $j = n$, we have

$$f_{1,n}(x_1, x_n) = r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} f(x_n),$$

$$-\infty < x_1 < x_2 < \infty.$$

The conditional distribution of $X_{U(j)} \mid X_{U(i)} = x_i$ is

$$f(X_{U(j)} \mid X_{U(i)} = x_i) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)}$$

$$= \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{\overline{F}(x_i)},$$

$$-\infty < x_i < x_j < \infty \tag{2.5}$$

and for $X_{U(i)} \mid X_{U(j)} = x_j$ is

$$f(X_{U(i)} \mid X_{U(j)} = x_j)$$

$$= \frac{(j-1)!}{(i-1)!(j-i-1)!} \left[\frac{R(x_i)}{R(x_j)}\right]^{i-1} \left[1 - \frac{R(x_i)}{R(x_j)}\right]^{j-i-1} \frac{r(x_i)}{R(x_j)},$$

$$-\infty < x_i < x_{i+1} < \infty. \tag{2.6}$$

3. Best Linear Unbiased Estimation

In life-testing experiments an experimenter may often have to terminate the experiment after a certain number of units failed instead of waiting for all the units to fail. This is naturally both time and cost effective. Samples observed in this manner are called Type-II censored samples. Best linear unbiased estimation (BLUE) is one of the commonly used methods of estimation for the scale or the location and the scale parameters of a population when the available sample is either complete or Type-II censored; see for example, Rao (1973), David (1981), Balakrishnan and Cohen (1991), and Arnold *et al.*(1992).

The best linear unbiased estimate (BLUE) of a parameter θ based on data X is

1. a linear function of $X' = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$. That is, the estimator can be written as $a'X$,

2. unbiased ($E[a'X]=\theta$), and

3. has the smallest variance among all unbiased linear estimators.

Suppose $X'=(X_{1:n},X_{2:n},\ldots,X_{n:n})$ denotes the vector of available order statistics from a scale distribution, that is, a distribution with probability density function of the form $\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$. Then the best linear unbiased estimator of the scale parameter σ is (Balakrishnan and Cohen, 1991)

$$\hat{\sigma}=\alpha'\Sigma^{-1}X/(\alpha'\Sigma^{-1}\alpha) \tag{3.1}$$

and its variance is

$$Var(\hat{\sigma})=\sigma^2/(\alpha'\Sigma^{-1}\alpha). \tag{3.2}$$

Here, α and Σ denote the mean vector and the variance-covariance matrix of the order statistics from the standard distribution, respectively.

Similarly, if X denotes the vector of available order statistics from a location and scale distribution

$$\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right),$$

then BLUEs of the location and scale parameters μ and σ are (Balakrishnan and Cohen, 1991)

$$\hat{\mu}=\frac{1}{\Delta}\{\alpha'\Sigma^{-1}\alpha 1'\Sigma^{-1}-\alpha'\Sigma^{-1}1\alpha'\Sigma^{-1}\}X \tag{3.3}$$

and

$$\hat{\sigma}=\frac{1}{\Delta}\{1'\Sigma^{-1}1\alpha'\Sigma^{-1}-1'\Sigma^{-1}\alpha 1'\Sigma^{-1}\}X \tag{3.4}$$

where

$$\Delta=(\alpha'\Sigma^{-1}\alpha)(1'\Sigma^{-1}1)-(\alpha'\Sigma^{-1}1)^2. \tag{3.5}$$

The variances and covariance of these estimators are given by

$$Var(\hat{\mu}) = \frac{\sigma^2(\alpha'\Sigma^{-1}\alpha)}{\Delta}, \tag{3.6}$$

$$Var(\hat{\sigma}) = \frac{\sigma^2(1'\Sigma^{-1}1)}{\Delta}, \tag{3.7}$$

and

$$Cov(\hat{\mu},\hat{\sigma}) = \frac{-\sigma^2(\alpha'\Sigma^{-1}1)}{\Delta}. \tag{3.8}$$

Although the above formulae may be used numerically to determine the BLUEs of the parameters for any distribution, only a few distributions allow explicit derivation of these estimators. Exponential and uniform distributions are commonly used as examples to illustrate this method of estimation under complete or Type-II censored samples; see, for example, David (1981), Balakrishnan and Cohen (1991), and Arnold *et al.* (1992).

In some cases (like the exponential distribution) the BLUEs are identical to the maximum likelihood estimators. In other cases also the BLUEs are not only explicit linear estimators, but are also highly efficient estimators; in fact, the BLUEs are asymptotically fully efficient as compared to the maximum likelihood estimators.

4. Best Linear Unbiased Prediction

In many life testing, reliability and replacement policy situations, it is desirable to predict the time of future failures from times of the early failures in the same sample. Best linear unbiased prediction (BLUP) is used in linear mixed models for the estimation of random effects. BLUP was derived by Charles Roy Henderson in 1950 but the term "best linear unbiased predictor" (or "prediction") seems not to have been used until 1962. "Best linear unbiased predictions" (BLUPs) of random effects are similar to best linear unbiased estimates (BLUEs) (see Gauss–Markov theorem) of fixed effects.

Goldberger (1962) and Whittle (1963) have made studies of BLUPs in the general linear model. Using results of Goldberger (1962) and Whittle (1963), we obtain the BLUPs of the "future" observation $X_{s:n}$ based on the observed values $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$, $1 \leq r < s \leq n$. For example, if n items are put into service simultaneously and items are to begin being replaced as soon as only 100(1 - λ^*) % of them remain functioning, then we can predict the replacement time $X_{s:n}$ (where $s \cong n\lambda^*$) from the early failure times $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$. As another example, if n items form an n-component parallel system, then we can predict the time of system failure, which is just $X_{n:n}$.

Except for a few well-behaved cases (the exponential, Pareto and a few other distributions), the quantities needed to compute the BLUE predictor of $X_{s:n}$ may be difficult or impossible to accumulate. Indeed, information on expectations and especially covariance of order statistics from many well-known failure distributions is either nonexistent or not widely available. Even when it is available, one is faced with inversion of $n \times n$ matrices. Fortunately, if the sample size is moderately large, we can appeal to sample quantile theory for asymptotically BLU (ABLU) prediction procedures. Here, the required inverses are completely known in terms of the underlying density f. Further impetus for using sample quantiles is gained from the well known fact that for many distributions, retaining only a few strategically spaced order statistics (sample quantiles) leads to highly efficient estimates (see David (1970), Harter (1969) or Sarhan (1962) for references). Prediction intervals in the above settings have been treated by Hewett (gamma) (1968), Lawless (exponential) (1973) and Kaminsky and Nelson (exponential) (1974).

We say that the linear combination $\hat{X}_{s:n}$ of the data $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$, $1 \le r < s \le n$ is the BLUP of $X_{s:n}$ *if* and only *if*

1. $E(X_{s:n} - \hat{X}_{s:n}) = 0$ and
2. $E(X_{s:n} - \hat{X}_{s:n})^2$ is a minimum.

It follows from the results of Goldberger (1962) and Whittle (1963) that the BLUP of $X_{s:n}$ is given by,

$$\hat{X}_{s:n} = \hat{E}(X_{s:n}) + W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha)$$

where $1' = (1, \cdots, 1)$ is $1 \times r$, $W' = (W_1, W_2, \ldots, W_r)$

$$W_i = Cov(X_{i:n}, X_{j:n}), \quad i, j = 1, 2, \ldots, r$$

$$Var - Cov(X) = \sigma^2 \Sigma.$$

and $\hat{\mu}$ and $\hat{\sigma}$ are BLUEs of μ and σ.

$$\hat{E}(X_{s:n}) = \hat{\mu} + \hat{\sigma}\alpha$$

is the BLUE of $E(X_{s:n})$.

Clearly $\hat{E}(X_{s:n})$ provides an unbiased predictor of $X_{s:n}$ but, of course, its mean squared error (*mse*) will exceed to that $X_{s:n}$. It is well known that the best (unrestricted) least-square predictor of $X_{s:n}$ is

$$\hat{X}_{s:n} = E(X_{s:n} \mid X_{1:n}, X_{2:n}, \ldots, X_{r:n})$$

but $\hat{X}_{s:n}$ in general depends on the unknown parameters. However, its *mse* does provide a lower bound for the error in predicting $X_{s:n}$.

5. Some Continuous Distributions

5.1: Weibull distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by

$$f(x) = \frac{\gamma(x-\mu)^{\gamma-1}}{\sigma^{\gamma}} e^{-\left(\frac{x-\mu}{\sigma}\right)^{\gamma}}, \mu < x < \infty, \sigma > 0 \qquad (5.1)$$

and the *df* is given by

$$F(x) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)^{\gamma}}, \qquad \mu < x < \infty, \sigma > 0. \qquad (5.2)$$

If we put $\gamma = 1$ in (5.1), we get the *pdf* of exponential distribution and for $\gamma = 2$, (5.1) gives *pdf* of Rayleigh distribution.

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

5.2: Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x,\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, x \geq \mu, \ \sigma > 0 \qquad (5.3)$$

and the *df* is given by

$$F(x) = 1 - e^{-(x-\mu)/\sigma}, \ x \geq \mu, \sigma > 0. \qquad (5.4)$$

USAGE: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption:

$$\Pr(X > s + t \mid X > s) = \Pr(X > t), \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

5.3: GENERALIZED PARETO DISTRIBUTION

A random variable X is said to have the generalized Pareto distribution if its *pdf* is of the following form

$$f(x,\mu,\sigma,\beta) = \begin{cases} \dfrac{1}{\sigma}\left\{1+\beta\left(\dfrac{x-\mu}{\sigma}\right)\right\}^{-(1+1/\beta)} & ,x \geq \mu, for\ \beta > 0, \\ & \mu < x < \mu - \sigma/\beta\ for\ \beta < 0, \\ \dfrac{1}{\sigma}e^{-(x-\mu)/\sigma} & ,x \geq \mu, \quad for\ \beta = 0, \\ 0 & ,\text{otherwise} \end{cases} \tag{5.5}$$

and the *df* is given by

$$F(x) = 1 - \left\{1+\beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-1/\beta}, \quad \beta \neq 0, \tag{5.6}$$

$$= 1 - e^{-(x-\mu)/\sigma}, \qquad \beta = 0.$$

For $\beta > 0$, generalized Pareto distribution is known as Pareto type II or Lomax distribution. For $\beta = -1$, this distribution coincide with the uniform distribution on $(\mu, \mu + \sigma)$.

USAGE: The generalized Pareto distribution was introduced by Pickands (1975). Some of its application includes its uses in the analysis of extreme events, in the modeling of large insurance claims and to describe the annual maximum flood at river gauging station. Generalized Pareto distribution has finite variance if $\beta < 1/2$.

5.4: POWER FUNCTION DISTRIBUTION

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below

$$f(x) = \frac{\gamma}{(\beta-\alpha)}\left(\frac{\beta-x}{\beta-\alpha}\right)^{\gamma-1}, \ \alpha \le x \le \beta, \ \gamma > 0 \tag{5.7}$$

$$F(x) = 1 - \left(\frac{\beta-x}{\beta-\alpha}\right)^{\gamma}, \ \alpha \le x \le \beta, \ \gamma > 0. \tag{5.8}$$

USAGE: The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. For $\gamma = 1$ (5.7) is the *pdf* of a two-parameter rectangular distribution. The consumption of fuel by an airplane during a flight may be assumed to be rectangular with parameters α and β. The thickness of steel produced by the rolling machines of steel plants may be considered as rectangular distribution with parameters α and β (Ahsanullah, 1986).

5.5: LOGISTIC DISTRIBUTION

The logistic distribution is described by the *pdf*

$$f(x;\mu,\sigma) = (1/\sigma')\exp\{-(x-\mu)/\sigma'\}\}/[1+\exp\{-(x-\mu)/\sigma'\}\}]^2,$$

$$-\infty < x < \infty, \qquad -\infty < \mu < \infty \tag{5.9}$$

where $\sigma' = \frac{\sqrt{3}\sigma}{\pi} > 0.$

This distribution is symmetrical with mean μ and variance σ^2.

5.6: UNIFORM DISTRIBUTION

A random variable X is said to have a uniform distribution (or rectangular distribution) if its *pdf* is given by

$$f(x) = \frac{1}{\sigma} \quad , \mu - \sigma/2 \le x \le \mu + \sigma/2, \; -\infty < \mu < \infty, \sigma > 0 \qquad (5.10)$$

$$= 0 \quad \text{, otherwise .}$$

It is noted that every distribution function $F(x)$ follows uniform distribution $U(0,1)$.

USAGE: This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

RECURRENCE RELATION FOR SINGLE AND PRODUCT MOMENTS OF ORDER STATISTICS AND RECORD VALUES

1. INTRODUCTION

Since means and variance-covariance matrices are needed for the BLUE and BLUP for order statistics and records, we have therefore discussed in this chapter recurrence relations for single and product moments of order statistics and record values. In section 2, general results for finding the k^{th} moment of the r^{th} order statistic and the product moment of the j^{th} power of the r^{th} order statistic and the k^{th} power of the s^{th} order statistic are obtained without considering any particular distribution. In section 3, these results are then utilized to obtain recurrence relations for doubly truncated and non-truncated distributions. In section 4, recurrence relations for single and product moments of record values for some continuous distributions are discussed. The examples under consideration are Weibull, exponential, Pareto, power function and logistic distributions. For more examples on linear exponential distribution and generalized exponential distribution one can refer to Saran and Pushkaran (1999 a, b; 2000).

2. MOMENTS FOR ORDER STATISTICS

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics obtained from a continuous *df* $F(x)$ and *pdf* $f(x)$. Then the *pdf* of $X_{r:n}$ $(1 \leq r \leq n)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x),$$

$$-\infty < x < \infty. \qquad (2.1)$$

Let $\alpha_{r:n}^{(k)} = E(X_{r:n}^{k})$, be the k^{th} moment of the r^{th} order statistic. Then

$$\alpha_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^k \{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x)\,dx \qquad (2.2)$$

The *pdf* of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by

$$c_{r,s:n}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s} f(x)f(y),$$

$$-\infty < x < y < \infty, \qquad (2.3)$$

where

$$c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Let

$$\alpha_{r,s:n}^{(j,k)} = E(X_{r:n}^{j}\, X_{s:n}^{k})$$

then

$$\alpha_{r,s:n}^{(j,k)} = c_{r,s:n}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^j y^k \{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s} f(x)f(y)\,dy\,dx. \qquad (2.4)$$

The *pdf* in case of truncation from both the sides is

$$\frac{f(x)}{(P-Q)}, \qquad Q_1 < x < P_1 \qquad (2.5)$$

where

$$\int_{-\infty}^{Q_1} f(x)\,dx = Q$$

and

$$\int_{P_1}^{\infty} f(x)\,dx = 1 - P\ . \tag{2.6}$$

P and Q are assumed to be known ($Q < P$) and Q_1 and P_1 are functions of Q and P, respectively. For simplicity, $f(x)$ and $F(x)$ are used for truncation case as well, then in case of truncation from both sides:

$$\alpha_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} x^k\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x)\,dx \tag{2.7}$$

and

$$\alpha_{1:n}^{(k)} = n\int_{Q_1}^{P_1} x^k\{1-F(x)\}^{n-1} f(x)\,dx. \tag{2.8}$$

For the product moments, we have

$$\alpha_{r,s:n}^{(j,k)} = c_{r,s:n}\int_{Q_1}^{P_1}\int_{x}^{P_1} x^j y^k\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s} f(x) f(y)\,dy\,dx. \tag{2.9}$$

2.1: RECURRENCE RELATIONS FOR THE SINGLE MOMENTS

THEOREM 2.1: (*Khan et al., 1983 a*)

For Q_1 finite, $n \geq 1$ and $k = 1, 2, \ldots,$

$$\alpha_{1:n}^{(k)} = Q_1^k + k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^n\,dx. \tag{2.10}$$

PROOF: Integrating (2.8) by parts, the above result is established.

THEOREM 2.2: (*Khan et al., 1983 a*)

For $2 \leq r \leq n$, $n \geq 2$, and $k = 1, 2, \ldots,$

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} k\int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}\,dx \tag{2.11}$$

under the assumptions

$$\lim_{x \to Q_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} = 0$$

and

$$\lim_{x \to P_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} = 0.$$

PROOF: Using (2.7) we have

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r} f(x) dx$$

$$- \frac{(n-1)!}{(r-2)!(n-r)!} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-2} \{1 - F(x)\}^{n-r} f(x) dx$$

$$= \binom{n-1}{r-1} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-2} \{1 - F(x)\}^{n-r}$$

$$\times \{nF(x) - (r-1)\} f(x) dx..$$

Let

$$h(x) = -\{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} \tag{2.12}$$

Differentiating both sides of equation (2.12) with respect to x, we get

$$h'(x) = \{F(x)\}^{r-2} \{1 - F(x)\}^{n-r} \{nF(x) - (r-1)\} f(x) \tag{2.13}$$

Thus,

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} \int_{Q_1}^{P_1} x^k h'(x) dx . \tag{2.14}$$

Integrating (2.14) by parts and putting the value of $h(x)$ from (2.12) we get the result.

Also it can be seen that

$$\alpha_{r:n}^{(k)}-\alpha_{r-1:n}^{(k)}=\binom{n}{r-1}k\int_{Q_1}^{P_1}x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}dx \quad (2.15)$$

and

$$\alpha_{r-1:n-1}^{(k)}-\alpha_{r-1:n}^{(k)}=\binom{n-1}{r-2}k\int_{Q_1}^{P_1}x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}dx. \quad (2.16)$$

2.2: RECURRENCE RELATIONS FOR THE PRODUCT MOMENTS

THEOREM 2.3: (*Khan et al., 1983 b*)

For, $1 \le r < s \le n$ and $j, k > 0$,

$$\alpha_{r,s:n}^{(j,k)}-\alpha_{r,s-1:n}^{(j,k)}=c_{r,s-1:n}^{*}\,k\int_{Q_1}^{P_1}\int_{x}^{P_1}x^{j}y^{k-1}\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+1}f(x)\,dy\,dx$$

where (2.17)

$$c_{r,s-1:n}^{*}=\frac{n!}{(r-1)!(s-r-1)!(n-s+1)!}=\frac{c_{r,s-1:n}}{(s-r-1)}.$$

PROOF: We have

$$\alpha_{r,s:n}^{(j,k)}-\alpha_{r,s-1:n}^{(j,k)}=c_{r,s-1:n}^{*}\int_{Q_1}^{P_1}\int_{x}^{P_1}x^{j}y^{k}\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-2}\{1-F(y)\}^{n-s}$$

$$\times\{(n-r)F(y)-(n-s+1)F(x)-(s-r-1)\}$$

$$\times f(x)f(y)\,dy\,dx. \quad (2.18)$$

Let $$h(x,y)=-\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+1}, \quad (2.19)$$

then $$\frac{\partial h(x,y)}{\partial y}=\{F(y)-F(x)\}^{s-r-2}\{1-F(y)\}^{n-s}$$

$$\times\{(n-r)F(y)-(n-s+1)F(x)-(s-r-1)\}f(y).$$

(2.20)

Putting the value of (2.20) in (2.18), we get

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} \int_{Q_1}^{P_1} x^j \{F(x)\}^{r-1} f(x)$$

$$\times \left\{ \int_x^{P_1} y^k \frac{\partial}{\partial y} h(x,y)\,dy \right\} dx. \qquad (2.21)$$

Now in view of (2.19),

$$\int_x^{P_1} y^k \frac{\partial}{\partial y} h(x,y)\,dy = k \int_x^{P_1} y^{k-1} \{F(y) - F(x)\}^{s-r-1}$$

$$\times \{1 - F(y)\}^{n-s+1} dy. \qquad (2.22)$$

Substituting (2.21) in (2.22), the required expression is obtained.

COROLLARY 2.1: For $1 \le r \le n-1$ and $j, k > 0$,

$$\alpha_{r,r+1:n}^{(j,k)} = \alpha_{r:n}^{(j+k)} + c_{r:n} k \int_{Q_1}^{P_1} \int_x^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{1 - F(y)\}^{n-r} f(x)\,dy\,dx. \qquad (2.23)$$

where

$$c_{r:n} = \frac{c_{r,r+1:n}}{(n-r)} = \frac{n!}{(r-1)!(n-r)!}.$$

PROOF: Putting $s = r+1$ in Theorem 2.3 and noting that

$$\alpha_{r,r:n}^{(j,k)} = E\left(X_{r:n}^{j} X_{r:n}^{k}\right) = E\left(X_{r:n}^{j+k}\right) = \alpha_{r:n}^{(j+k)} \qquad (2.24)$$

we get the desired result.

COROLLARY 2.2: For $n > 1$ and $j, k > 0$,

$$\alpha_{n-1,n:n}^{(j,k)} = \alpha_{n-1:n}^{(j+k)} + n(n-1)k \int_{Q_1}^{P_1} \int_x^{P_1} x^j y^{k-1} \{F(x)\}^{n-2}$$

$$\times \{1 - F(y)\} f(x)\,dy\,dx.. \qquad (2.25)$$

PROOF: Put $r = n-1$ in Theorem 2.3, to get the result.

THEOREM 2.4: (*Khan et al., 1983 b*)

For $1 \le r < s \le n$ and $j > 0$,

$$\alpha^{(j,0)}_{r,s:n} = \alpha^{(j,0)}_{r,s-1:n} = \ldots \alpha^{(j,0)}_{r,r+1:n} = \alpha^{(j)}_{r:n}. \tag{2.26}$$

PROOF: From relation (2.18) and (2.21), with $k = 0$

$$\alpha^{(j,0)}_{r,s:n} = E(X^{j}_{r:n} X^{0}_{s:n}) = E(X^{j}_{r:n}) = \alpha^{(j)}_{r:n}.$$

3. RECURRENCE RELATIONS FOR SOME EXPLICIT DISTRIBUTIONS

Now we will use the results of Theorems 2.1, 2.2 and 2.3 to obtain the recurrences relations for some specific distributions. It is assumed throughout that

$$\alpha^{(0)}_{r:n} = 1, \qquad 1 \le r \le n,$$

$$\alpha^{(k)}_{0:n} = Q_1^k, \qquad k = 1, 2, \ldots,$$

$$\alpha^{(k)}_{n+1:n} = P_1^k$$

In fact starting with $\alpha^{(k)}_{1:n}$, $k = 1, 2, \ldots$, we can show that all the raw single moments of order statistics can be obtained systematically.

3.1: DOUBLY TRUNCATED WEIBULL AND EXPONENTIAL DISTRIBUTIONS

The *pdf* of doubly truncated distribution is (*Khan et al., 1983 a*)

$$f(x) = \frac{p\, x^{p-1} e^{-x^p}}{P-Q}, \quad -\log(1-Q) \le x^p \le -\log(1-P), \quad p > 0.$$

Here $\quad Q_1^p = -\log(1-Q) \quad$ and $\quad P_1^p = -\log(1-P)$.

Let $\quad Q_2 = (1-Q)/(P-Q) \quad$ and $\quad P_2 = (1-P)/(P-Q)$

then, $$\{1-F(x)\}=-P_2+\frac{1}{p}x^{1-p}f(x). \tag{3.1}$$

The moment relationship is given by

$$\alpha_{r:n}^{(k)}=Q_2\,\alpha_{r-1:n-1}^{(k)}-P_2\,\alpha_{r:n-1}^{(k)}+\frac{k}{np}\alpha_{r:n}^{(k-p)}. \tag{3.2}$$

For $r=n$, from (2.11) we get

$$\alpha_{n:n}^{(k)}=Q_2\,\alpha_{n-1:n-1}^{(k)}-P_2\,P_1^k+\frac{k}{np}\alpha_{n:n}^{(k-p)}.$$

For product moment, putting the value of $\{1-F(y)\}$ in (2.17), we get

$$\alpha_{r,s:n}^{(j,k)}=\alpha_{r,s-1:n}^{(j,k)}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}^{(j,k)}-\alpha_{r,s-1:n-1}^{(j,k)}\right)+\frac{k}{p(n-s+1)}\alpha_{r,s:n}^{(j,k-p)},$$

$$1\le r<s\le n,\ s-r\ge 2. \tag{3.3}$$

If we put $j=0$ in (3.3) then after noting that

$$\alpha_{r,s:n}^{(0,k)}=\alpha_{s:n}^{(k)}$$

we get

$$\alpha_{s:n}^{(k)}=\alpha_{s-1:n}^{(k)}-\frac{nP_2}{n-s+1}\left(\alpha_{s:n-1}^{(k)}-\alpha_{s-1:n-1}^{(k)}\right)+\frac{k}{p(n-s+1)}\alpha_{s:n}^{(k-p)} \tag{3.4}$$

a relation established for single moments by Khan *et al.*(1983 a).

For $s=r+1$, (3.3) reduces to

$$\alpha_{r,r+1:n}^{(j,k)}=\alpha_{r:n}^{(j+k)}-\frac{nP_2}{(n-r)}\left(\alpha_{r,r+1:n-1}^{(j,k)}-\alpha_{r:n-1}^{(j+k)}\right)+\frac{k}{p(n-r)}\alpha_{r,r+1:n}^{(j,k-p)},$$

$$(1\le r\le n-1),\,n\ge 3. \tag{3.5}$$

For $r=n-1$ and $s=n$, (3.3) reduces to

$$\alpha_{n-1,n:n}^{(j,k)}=\alpha_{n-1:n}^{(j+k)}-nP_2\left(P_1^k\,\alpha_{n-1:n-1}^{(j)}-\alpha_{n-1:n-1}^{(j+k)}\right)+\frac{k}{p}\alpha_{n-1,n:n}^{(j,k-p)},$$

$$n\ge 2. \tag{3.6}$$

If we put $p=1$ in the above expressions, we get corresponding results for the exponential distribution. For the non-truncated case, one has to put $P=1$, $Q=0$. In case of doubly truncated Weibull distribution, recurrence relations for $\alpha_{r:n}^{(k)}$ are available in Khan *et al.* (1983 a). Expressions for exact and explicit product moment with $j=k=1$ can be obtained in Leiblein (1955).

In case of $j=k=1$, for the Weibull distribution,

$$\alpha_{r,s:n}=\alpha_{r,s-1:n}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}-\alpha_{r,s-1:n-1}\right)+\frac{1}{p(n-s+1)}\alpha_{r,s:n}^{(1,1-p)},$$

$$1\le r<s\le n,\quad s-r\ge 2. \qquad (3.7)$$

For the exponential distribution, this reduces to

$$\alpha_{r,s:n}=\alpha_{r,s-1:n}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}-\alpha_{r,s-1:n-1}\right)+\frac{1}{(n-s+1)}\alpha_{r:n},$$

$$1\le r<s\le n,\quad s-r\ge 2. \qquad (3.8)$$

3.2: DOUBLY TRUNCATED POWER FUNCTION DISTRIBUTION

The *pdf* of doubly truncated power function distribution is given by (*Khan et al., 1983 a*)

$$f(x)=\frac{\nu a^{-\nu}x^{\nu-1}}{P-Q},\qquad aQ^{1/\nu}\le x\le aP^{1/\nu},\qquad a,\nu>0.$$

Here $Q_1=aQ^{1/\nu}$, and $P_1=aP^{1/\nu}$.

Let $Q_2=Q/(P-Q)$ and $P_2=P/(P-Q)$,

then

$$\{1-F(x)\}=P_2-\frac{x}{\nu}f(x). \qquad (3.9)$$

For the r^{th} order statistic,

$$\alpha_{r:n}^{(k)} = \left\{ P_2\, \alpha_{r:n-1}^{(k)} - Q_2 \alpha_{r-1:n-1}^{(k)} \right\} \frac{n\nu}{n\nu + k}. \tag{3.10}$$

Also for $r = n$ in view of (2.10), we get

$$\alpha_{n:n}^{(k)} = \left\{ P_2\, P_1^k - Q_2\, \alpha_{n-1:n-1}^{k} \right\} \frac{n\nu}{n\nu + k}. \tag{3.11}$$

For product moments, putting the value of $\{1 - F(y)\}$ from (3.9) in Theorem 2.3, we get the recurrence relation

$$\alpha_{r,s:n}^{(j,k)} = \frac{\nu}{\nu(n-s+1)+k} \left((n-s+1)\alpha_{r,s-1:n}^{(j,k)} + n\,P_2 \left(\alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)} \right) \right),$$

$$1 \le r < s \le n, \quad s - r \ge 2. \tag{3.12}$$

For $s = r+1$, we get from Corollary 2.1,

$$\alpha_{r,r+1:n}^{(j,k)} = \frac{\nu}{\nu(n-r)+k} \left((n-r)\alpha_{r:n}^{(j+k)} - n\,P_2 \left(\alpha_{r,r+1:n-1}^{(j,k)} - \alpha_{r:n-1}^{(j+k)} \right) \right),$$

$$1 \le r \le n-2, \quad n \ge 3 \tag{3.13}$$

after noting that $\alpha_{r,\,r:n}^{(j,k)} = \alpha_{r:n}^{(j+k)}$.

Similarly for $n = s = r+1$, we get

$$\alpha_{n-1,n:n}^{(j,k)} = \frac{\nu}{\nu + k} \left(\alpha_{n-1:n}^{(j+k)} + n\,P_2 \left(P_1^k\, \alpha_{n-1:n-1}^{(j)} - \alpha_{n-1:n-1}^{(j+k)} \right) \right), \; n \ge 2$$

$$\tag{3.14}$$

interpreting $\alpha_{n-1.n:n-1}^{(j,k)} = P_1^k\, \alpha_{n-1:n-1}^{(j)}$ as discussed in Section 3.1. Recurrence relations for single moments can be obtained by putting $j = 0$ as given in Khan *et al.* (1983 a).

For $j = k = 1$, the relations are available in Balakrishnan and Joshi (1981 b). The non-truncated cases $(P = 1, Q = 0)$ are discussed by Malik (1967). Reference may also be made to Khan *et al.* (1983 a) for the recurrence relations of $\alpha_{r:n}^{(l)}$, $l = 1, 2, \ldots$.

3.3: DOUBLY TRUNCATED PARETO DISTRIBUTIONS

The *pdf* of doubly truncated Pareto distribution is (*Khan et al, 1983 a*)

$$f(x) = \frac{v a^v x^{-v-1}}{P-Q}, \quad a(1-Q)^{-1/v} \le x \le a(1-P)^{-1/v}, \quad a,v > 0.$$

Here $\quad Q_1 = a(1-Q)^{-1/v} \quad$ and $\quad P_1 = a(1-P)^{-1/v}$.

Set $\quad Q_2 = (Q-1)/(P-Q) \quad$ and $\quad P_2 = (P-1)/(P-Q)$,

then,

$$\{1-F(x)\} = (x/v) f(x) + P_2. \tag{3.15}$$

For the r^{th} order statistic, $2 \le r \le n-1$,

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \frac{k}{nv}\alpha_{r:n}^{(k)} + P_2\left(\frac{n-1}{n-r}\right)\left\{\alpha_{r:n-1}^{(k)} - \alpha_{r-1:n-2}^{(k)}\right\}.$$

Using the recurrence relation given in (3.2) and simplifying, we get

$$(nv-k)\alpha_{r:n}^{(k)} = (P_2\,\alpha_{r:n-1}^{(k)} - Q_2\,\alpha_{r-1:n-1}^{(k)})nv. \tag{3.16}$$

And for $r = n$ from (2.11), it can be seen that

$$(nv-k)\alpha_{n:n}^{(k)} = (P_2 P_1^k - Q_2\,\alpha_{n-1:n-1}^{(k)})nv, \quad nv \ne k. \tag{3.17}$$

In view of (3.15) and Theorem 2.3, we get the recurrence relation for product moments,

$$\begin{aligned}(v(n-s+1)-k)&\alpha_{r,s:n}^{(j,k)}\\ &= v\left((n-s+1)\alpha_{r,s-1:n}^{(j,k)} + nP_2\left(\alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)}\right)\right)\end{aligned} \tag{3.18}$$

$$1 \le r < s \le n, \quad s-r \ge 2 \text{ and } v(n-s+1) \ne k.$$

However, if $k = v\,(n-s+1)$, we get from (3.18),

$$(n-s+1)\,\alpha_{r,s-1:n}^{(j,k)} = n\,P_2\left(\alpha_{r,s-1:n-1}^{(j,k)} - \alpha_{r,s:n-1}^{(j,k)}\right). \tag{3.19}$$

Marginal results for $k \ne v\,(n-s+1)$ can easily be seen to be equal to

$$(v(n-r)+k)\alpha_{r,r+1:n}^{(j,k)} = v\,((n-r)\alpha_{r:n}^{(j+k)} + n\,P_2(\alpha_{r,r+1:n-1}^{(j,k)} - \alpha_{r:n-1}^{(j+k)}))$$

$$1 \le r \le n-2\,, n \ge 3 \qquad (3.20)$$

$$(v-k)\alpha_{n-1,n:n}^{(j,k)} = v\,(\alpha_{n-1:n}^{(j+k)} + n\,P_2(P_1^k\,\alpha_{n-1:n-1}^{(j)} - \alpha_{n-1:n-1}^{(j+k)}))\,, \qquad n \ge 2. \qquad (3.21)$$

Recurrence relations for $j = k = 1$ have been obtained by Balakrishnan and Joshi (1982). Malik (1966) has obtained these results for $P = 1$, $Q = 0$. To evaluate $\alpha_{r,s:n}^{(j,k)}$, one may require the recurrence relations for $\alpha_{r:n}^{(l)}$ for which we refer to Khan *et al.* (1983 a) which can also be obtained by putting $j = 0$ and replacing s by r in this section.

3.4: DOUBLY TRUNCATED LOGISTIC DISTRIBUTION

The truncated *pdf* for logistic distribution is given by

(*Khan et al., 1983 a*)

$$f(x) = \frac{e^{-x}}{(P-Q)(1+e^{-x})^2}, \qquad Q_1 \le x \le P_1,$$

where

$$Q_1 = \log[Q/(1-Q)] \qquad \text{and} \qquad P_1 = \log[P/(1-P)]$$

and the *df* is

$$F(x) = \frac{(1+e^{-x})^{-1} - Q}{P-Q},$$

$$[1-F(x)] = \frac{P-(1+e^{-x})^{-1}}{P-Q}.$$

Thus, in the case of symmetric truncation $(Q = 1-P)$, we have

$$F(x)[1-F(x)] = -\frac{PQ}{(P-Q)^2} + \frac{P+Q}{P-Q}f(x). \qquad (3.22)$$

Thus, in view of (2.11) and (3.29)

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)}$$

$$= \binom{n-1}{r-1} k \int_{Q_1}^{P_1} x^{k-1} [F(x)]^{r-2} [1-F(x)]^{n-r}$$

$$\times \left\{ -\frac{PQ}{(P-Q)^2} + \frac{P+Q}{P-Q} f(x) \right\} dx$$

$$= -\frac{n-r}{r-1} \frac{PQ}{(P-Q)^2} [\, \alpha_{r-1:n-1}^{(k)} - \alpha_{r-2:n-2}^{(k)}]$$

$$+ \frac{P+Q}{(P-Q)} \frac{k}{r-1} \alpha_{r-1:n-1}^{(k-1)}.$$

That is,

$$\alpha_{r:n}^{(k)} = \left[1 - \frac{n-1}{r-1} \frac{PQ}{(P-Q)^2} \right] \alpha_{r-1:n-1}^{(k)} + \frac{n-1}{r-1} \frac{PQ}{(P-Q)^2} \alpha_{r-2:n-2}^{(k)}$$

$$+ \frac{P+Q}{P-Q} \frac{k}{r-1} \alpha_{r-1:n-1}^{(k)}. \tag{3.23}$$

If we set $P=1$ and $Q=0$ in (3.23), we get

$$\alpha_{r:n}^{(k)} = \alpha_{r-1:n-1}^{(k)} + \frac{k}{r-1} \alpha_{r-1:n-1}^{(k-1)} \tag{3.24}$$

This expression for non-truncated logistic distributions was obtained by Shah (1970). For non-symmetric truncation refer to Alshboul and Khan (1989).

4. Moments Of Record Values

The moment of the n^{th} record is

$$E(X_{U(n)}^k) = \int x^k \frac{R^{n-1}(x)}{(n-1)!} f(x) dx. \tag{4.1}$$

where $R(x) = -\log \overline{F}(x)$ as defined in chapter 1.

And the product moments

$$E(X_{U(m)}^{j} X_{U(n)}^{k}) = \iint x^{j} x^{k} f_{j,k}(x_j, x_k) dx_j \, dx_k \tag{4.2}$$

where $f_{j,k}(x_j, x_k) = \dfrac{(R(x_j))^{j-1}}{(j-1)!} r(x_j) \dfrac{(R(x_k) - R(x_j))^{k-j-1}}{(k-j-1)!} f(x_k)$.

4.1: RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS FOREXPONENTIA L DISTRIBUTION

Let X be a continuous random variable distributed exponentially with parameters μ and σ and the *pdf*

$$f(x, \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, x \geq \mu, \sigma > 0.$$

Then the *pdf* for the n^{th} upper record value $X_{U(n)}$ is given as

$$f_n(x) = \frac{1}{\sigma^n (n-1)!} (x-\mu)^{n-1} e^{-(x-\mu)/\sigma}, x \geq \mu. \tag{4.3}$$

And the joint *pdf* of $X_{U(m)}$ and $X_{U(n)}, n > m$ is

$$f_{m,n}(x, y) = \frac{1}{\sigma^n (m-1)!} \frac{(x-\mu)^{m-1}}{(n-m-1)!} (y-x)^{n-m-1} e^{-(y-\mu)/\sigma},$$

$$\mu \leq x < y < \infty. \tag{4.4}$$

THEOREM 4.1: (*Ahsanullah, 2004*)

For $n \geq 1$ and $r = 0, 1, 2, \ldots$

$$E(X_{U(n)}^{r+1}) = E(X_{U(n-1)}^{r+1}) + (r+1) E(X_{U(n)}^{r}) \tag{4.5}$$

and consequently, for $0 \leq m \leq n-1$ we can write

$$E(X_{U(n)}^{r+1}) = E(X_{U(m)}^{r+1}) + (r+1) \sum_{i=m+1}^{n} E(X_{U(i)}^{r+1}) \tag{4.6}$$

with $E(X_{U(0)}^{r+1}) = 0$ and $E(X_{U(n)}^{0}) = 1$.

THEOREM 4.2: (*Ahsanullah, 2004*)

For $n \geq 1$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(n)}^{r} X_{U(n+1)}^{s+1}) = E(X_{U(n)}^{r+s+1}) + (s+1)\, E(X_{U(n)}^{r} X_{U(n+1)}^{s}) \tag{4.7}$$

and for $1 \leq m \leq n-2, r, s = 0, 1, 2, \ldots$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1})$$

$$= E(X_{U(m)}^{r} X_{U(n-1)}^{s+1}) + (s+1) E(X_{U(m)}^{r} X_{U(n)}^{s}). \tag{4.8}$$

REMARK: By repeated application of the recurrence relation in (4.7) and (4.8), we obtain for $n \geq m+1$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1})$$

$$= E(X_{U(m)}^{r+s+1}) + (s+1) \sum_{i=m+1}^{n} E(X_{U(m)}^{r} X_{U(i)}^{s}).$$

THEOREM 4.3: (*Ahsanullah, 2004*)

For $n \geq 2$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(n-1)}^{r+1} X_{U(n)}^{s})$$

$$= E(X_{U(n)}^{r+s+1}) - (r+1)\, E(X_{U(n)}^{r} X_{U(n+1)}^{s}) \tag{4.9}$$

and for $2 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$

$$E(X_{U(m-1)}^{r+1} X_{U(n-1)}^{s})$$

$$= E(X_{U(m)}^{r+1} X_{U(n-1)}^{s}) - (r+1)\, E(X_{U(m)}^{r} X_{U(m+1)}^{s}). \tag{4.10}$$

REMARK: By repeated application of the recurrence relation in (4.9) and (4.10), we obtain for $n \geq 2$

$$E(X_{U(n-1)}^{r+1} X_{U(n)}^{s}) = \sum_{i=0}^{r+1} (-1)^i (r+1)^{(i)} E(X_{U(n+i)}^{r+s+1-i})$$

and for $2 \le m \le n-2$

$$E(X_{U(m-1)}^{r+1} X_{U(n-1)}^{s})$$

$$= \sum_{i=0}^{r+1} (-1)^p (r+1)^{(p)} E(X_{U(n-i)}^{r+1-i} X_{U(n+1-i)}^{s}).$$

4.2: RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS FOR GENERALIZED PARETO DISTRIBUTION

A random variable X is said to have the generalized Pareto distribution if its *pdf* is of the following form

$$f(x,\mu,\sigma,\beta) = \frac{1}{\sigma}\left\{1+\beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-(1+1/\beta)} \quad , x \ge \mu, for\ \beta > 0,$$

$$\mu < x < \mu - \sigma/\beta\ for\ \beta < 0. \qquad (4.11)$$

The *pdf* of $X_{U(n)}$ is

$$f_n(x) = \frac{1}{(n-1)!}\left\{\frac{1}{\beta}\ln\left[1+\frac{\beta(x-\mu)}{\sigma}\right]\right\}^{n-1} \frac{1}{\sigma}\left[1+\frac{\beta(x-\mu)}{\sigma}\right]^{-(1+1/\beta)}$$

$$\mu < x < \infty, \beta > 0.$$

THEOREM 4.4: (*Ahsanullah, 2004*)

For $n \ge 1$ and $r = 0,1,2,\ldots$

$$E(X_{U(n)}^{r+1}) = \frac{1}{1-(r+1)\beta}\{(r+1)E(X_{U(n)}^{r}) + E(X_{U(n-1)}^{r+1})\},$$

$$\beta < (r+1)^{-1}. \qquad (4.12)$$

REMARK: The recurrence relation in (4.12) can be used in a simple recursive manner to compute all the single moments of all record values. By setting $r=0$ in (4.12), we get the relation

$$E(X_{U(n)}) = \frac{1}{1-\beta}\{1 + E(X_{U(n-1)})\}, n \geq 2, \beta < 1. \tag{4.13}$$

Repeated application of (4.13) will readily yield

$$E(X_{U(n)}) = \frac{1}{1-\beta} + \frac{1}{(1-\beta)^2} + \ldots + \frac{1}{(1-\beta)^n}$$

$$= \frac{1}{\beta}\left[\frac{1}{(1-\beta)^n} - 1\right] \tag{4.14}$$

an expression given by Ahsanullah (1992).

THEOREM 4.5: (*Ahsanullah, 2004*)

For $n \geq 1$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(n)}^{r} X_{U(n+1)}^{s+1})$$

$$= \frac{1}{1-(s+1)\beta}[(s+1)\ E(X_{U(n)}^{r} X_{U(n+1)}^{s}) + E(X_{U(n)}^{r+s+1})] \tag{4.15}$$

for $\beta < \frac{1}{(s+1)}$.

And for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1})$$

$$= \frac{1}{1-(s+1)\beta}[(s+1)\ E(X_{U(m)}^{r} X_{U(n)}^{s}) + E(X_{U(m)}^{r} X_{U(n-1)}^{s+1})] \tag{4.16}$$

for $\beta < \frac{1}{(s+1)}$.

REMARKS

1. The recurrence relation in (4.16) can be used in a simple recursive manner to compute all the product moments of all record values. It is

known that the generalized Pareto distribution in (4.11) has finite variance if $\beta < 1/2$. In this case, by setting $r = 1$ and $s = 0$ in (4.13) we get

$$E(X_{U(n)} X_{U(n+1)}) = \frac{1}{1-\beta}[\ E(X_{U(n)}) + E(X^2_{U(n)})].$$

Similarly by setting $r = 1$ and $s = 0$ in (4.14) we get for $n < m + 2$

$$E(X_{U(m)} X_{U(n)}) = \frac{1}{1-\beta}[\ E(X_{U(m)}) + E(X_{U(m)} X_{U(n-1)})].$$

2. Upon letting the shape parameter $\beta \to 0$ in the recurrence relations presented in (4.9), (4.13), (4.15) and (4.16), we simply deduce the relations for the single and product moments of upper record values from the standard exponential distribution.

4.3: RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS FOR POWER FUNCTION DISTRIBUTION

A random variable X is said to have a power function distribution if its *pdf* is of the form given below:

$$f(x) = \frac{\gamma}{(\beta-\alpha)}\left(\frac{\beta - x}{\beta-\alpha}\right)^{\gamma-1},\ \alpha \le x \le \beta,\ \ \gamma > 0.$$

The marginal *pdf* of $X_{U(n)}$ is

$$f_n(x) = \frac{\gamma^n}{(n-1)!}(\beta-\alpha)^{-\gamma}(\beta-x)^{\gamma-1}[\ln(\beta-\alpha) - \ln(\beta - x)]^{n-1},$$

$$\alpha \le x \le \beta.$$

THEOREM 4.6: (*Ahsanullah, 2004, Saran and Pandey, 2004*)

For $n \ge 2$ and $r = 0,1,2,\ldots$

$$E(X^{r+1}_{U(n)}) = \frac{r+1}{\gamma+r+1}E(X^r_{U(n)}) + \frac{\gamma}{\gamma+r+1}E(X^{r+1}_{U(n-1)}). \tag{4.17}$$

REMARK: By repeatedly applying the recurrence relation in (4.17), we get for $n > 2$, $1 < m < n-1$ and $r = 0,1,2,\dots$

$$E(X_{U(n)}^{r+1}) = \left(\frac{r+1}{\gamma+r+1}\right)^{n-m-1} \sum_{i=0} \left(\frac{\gamma}{\gamma+r+1}\right)^{p} E(X_{U(n-i)}^{r})$$

$$+\left(\frac{\gamma}{\gamma+r+1}\right)^{n-m} E(X_{U(m)}^{r+1}). \tag{4.18}$$

THEOREM 4.7: (*Ahsanullah, 2004, Saran and Pandey, 2004*)

For $n \geq 1$ and $r, s = 0,1,2,\dots$

$$E(X_{U(n)}^{r} X_{U(n+1)}^{s+1}) = \frac{s+1}{\gamma+s+1} E(X_{U(n)}^{r} X_{U(n+1)}^{s})$$

$$+\frac{\gamma}{\gamma+s+1} E(X_{U(n)}^{r+s+1}) \tag{4.19}$$

and for $1 \leq m \leq n-2$, $r, s = 0,1,2,\dots$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1}) = \frac{s+1}{\gamma+s+1} E(X_{U(m)}^{r} X_{U(n)}^{s})$$

$$+\frac{\gamma}{\gamma+s+1} E(X_{U(m)}^{r} X_{U(n-1)}^{s+1}). \tag{4.20}$$

REMARK: By repeated application of the recurrence relation in (4.19) and (4.20), we get from $n \geq 1$ and $r, s = 0,1,2,\dots$

$$E(X_{U(n)}^{r} X_{U(n+1)}^{s+1}) = \gamma \sum_{i=0}^{s+1} \frac{(s+1)^{(i)}}{(\gamma+s+1+i)^{(i+1)}} E(X_{U(n)}^{r+s+1})$$

and for $1 \leq m \leq n-2$, $r, s = 0,1,2,\dots$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1}) = \gamma \sum_{i=0}^{s+1} \frac{(s+1)^{(i)}}{(\gamma+s+1+i)^{(i+1)}} E(X_{U(m)}^{r} X_{U(n-1)}^{s+1-i}).$$

THEOREM 4.8: (*Ahsanullah, 2004*)

For $n > 2$ and $r,s = 0,1,2,\ldots$

$$E(X_{U(n)}^{r+1} X_{U(n+1)}^{s}) = \frac{\gamma}{r+1}\{E(X_{U(n)}^{r+s+1}) - E(X_{U(n-1)}^{r+1} X_{U(n)}^{s})\}$$
$$- E(X_{U(n)}^{r} X_{U(n+1)}^{s}) \qquad (4.21)$$

and for $2 \le m \le n-2$, $r,s = 0,1,2,\ldots$

$$E(X_{U(m)}^{r+1} X_{U(n)}^{s}) = \frac{\gamma}{r+1}\{E(X_{U(m)}^{r+1} X_{U(n-1)}^{s}) - E(X_{U(m-1)}^{r+1} X_{U(n-1)}^{s})\}$$
$$- E(X_{U(m)}^{r} X_{U(n)}^{s}). \qquad (4.22)$$

4.4: RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS FOR UNIFORM DISTRIBUTION

Let $\{X_n, n \ge 1\}$ be a sequence of *i.i.d.* random variables from a uniform distribution with the following *pdf*

$$f(x) = \frac{1}{\sigma}, \quad \mu - \frac{\sigma}{2} < x < \mu + \frac{\sigma}{2}.$$

The *pdf* $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{1}{(n-1)!}\frac{1}{\sigma}\left\{-\ln\left(\frac{2\mu + \sigma - 2x}{2\sigma}\right)\right\}^{n-1}, \quad \mu - \frac{\sigma}{2} < x < \mu + \frac{\sigma}{2}.$$

THEOREM 4.9: (*Ahsanullah, 2004*)

We will assume here without any loss of generality $\sigma = 1$. For $n \ge 2$, and $r = 0,1,2,\ldots.$

$$E(X_{U(n)}^{r+1}) = \frac{r+1}{r+2}E(X_{U(n)}^{r}) + \frac{1}{r+2}E(X_{U(n-1)}^{r+1}). \qquad (4.23)$$

REMARK: By repeatedly applying the recurrence relation in (4.23), we get for $n \ge 2$, $1 \le m \le n-1$ and $r = 0,1,2,\ldots.$

$$E(X_{U(n)}^{r+1}) = \frac{r+1}{r+2}\sum_{i=0}^{n-m-1}\left(\frac{1}{r+2}\right)^{i} E(X_{U(n-i)}^{r}) + \left(\frac{1}{r+2}\right)^{n-m} E(X_{U(m)}^{r+1}).$$

(4.24)

THEOREM 4.10: (*Ahsanullah, 2004*)

For $n \geq 1$ and $s, r = 0,1,2,\ldots.$

$$E(X_{U(n)}^{r} X_{U(n+1)}^{s+1}) = \frac{s+1}{s+2} E(X_{U(n)}^{r} X_{U(n+1)}^{s}) + \frac{1}{s+2} E(X_{U(n)}^{r+s+1}).$$

(4.25)

And for $1 \leq m \leq n-2$ and $s, r = 0,1,2,\ldots.,$

$$E(X_{U(m)}^{r} X_{U(n)}^{s+1}) = \frac{s+1}{s+2} E(X_{U(m)}^{r} X_{U(n)}^{s}) + \frac{1}{s+2} E(X_{U(m)}^{r} X_{U(n-1)}^{s+1}).$$

(4.26)

REMARK: For $n \geq 1$

$$Cov(X_{U(n)}, X_{U(n+1)}) = \frac{1}{2} Var(X_{U(n)}) \quad (4.27)$$

$$Cov(X_{U(m)}, X_{U(n)}) = \frac{1}{2} Cov(X_{U(n)} X_{U(n-1)}). \quad (4.28)$$

Consequently, for $1 \leq m \leq n-1$

$$Cov(X_{U(m)}, X_{U(n)}) = \left(\frac{1}{2}\right)^{n-m} Var(X_{U(n)}). \quad (4.29)$$

CHAPTER III

BEST LINEAR UNBIASED ESTIMATORS OF ORDER STATISTICS

1. INTRODUCTION

In this chapter, BLUEs of scale and location parameters for exponential, generalized Pareto, logistic and uniform distributions have been considered using order statistics. The variance and covariance of the parameters are also discussed here.

2. BEST LINEAR UNBIASED ESTIMATORS FOR CONTINUOUS DISTRIBUTION (*Balakrishnan and Cohen, 1991*)

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the absolutely continuous *cdf*, where μ is the location parameter and $\sigma > 0$ is the scale parameter. Let Y denote the vector of the order statistics in the sample. Here the estimators of μ and σ which are the best among the unbiased linear functions of the components of Y are obtained. The procedure is based on the least-square theory originally developed by Aitken (1935). Llyod (1952) first applied the general results in order statistics.

Let $X = (Y - \mu)/\sigma$ be the standardized population random variable with *cdf* $F_0(x) = F(x:0,1)$. Clearly F_0 is free of the parameters and, hence the means and the covariances of the order statistics from the X population, $\mu_{i:n}$ and Σ, are free of them as well. Let X denote the vector of X-order statistics corresponding to Y. Then, it is clear that

$$E(Y_{i:n}) = \mu + \sigma \mu_{i:n} \tag{2.1}$$

and

$$Cov(Y_{i:n}, Y_{j:n}) = \sigma^2 \sigma_{i,j:n}, \tag{2.2}$$

for $1 \le i, j \le n$.

Let α be the mean vector of X, and $\theta' = (\mu, \sigma)$ be the vector of the unknown parameters. Further, let 1 be an $n \times 1$ vector whose components are all 1's. Then the n equations in (2.1) can be expressed in the matrix form as

$$E(Y) = A\theta \tag{2.3}$$

where the $n \times 2$ matrix $A = (1, \alpha)$ is completely specified. Also, (2.2) can be put in the form

$$Cov(Y) = \sigma^2 \Sigma \tag{2.4}$$

where $Cov(Y)$ represents the covariance matrix of Y, and the covariance matrix Σ of X, is known.

Suppose the goal is to choose μ and σ so that the quadratic form

$$\begin{aligned} Q(\theta) &= (Y - A\theta)'\Sigma^{-1}(Y - A\theta) \\ &= (Y - \mu 1 - \sigma\alpha)'\Sigma^{-1}(Y - \mu 1 - \sigma\alpha) \\ &= Y'\Sigma^{-1}Y - 2\mu 1'\Sigma^{-1}Y - 2\sigma\alpha'\Sigma^{-1}Y + 2\mu\sigma\alpha'\Sigma^{-1}1 \\ &\quad + \mu^2 1'\Sigma^{-1}1 + \sigma^2\alpha'\Sigma^{-1}\alpha \end{aligned} \tag{2.5}$$

is minimized.

If $Q(\theta)$ is minimized when $\theta = \hat{\theta} = (\hat{\mu}, \hat{\sigma})'$, then we say that $\hat{\mu}$ and $\hat{\sigma}$ are the *Best linear unbiased estimators* (BLUEs) of μ and σ, respectively.

On differentiating (2.5) with respect to μ and σ and equating to 0, we obtain the normal equations as

$$(1'\Sigma^{-1}1)\mu + (\alpha'\Sigma^{-1}1)\sigma = 1'\Sigma^{-1}Y,$$

$$(\alpha'\Sigma^{-1}1)\mu + (\alpha'\Sigma^{-1}\alpha)\sigma = \alpha'\Sigma^{-1}Y.$$

On solving these equations for μ and σ we obtain the solution

$$\hat{\mu} = -\alpha'\Gamma Y \qquad \text{and} \qquad \hat{\sigma} = 1'\Gamma Y \tag{2.6}$$

where $\Gamma = \Sigma^{-1}(1\alpha' - \alpha 1')\Sigma^{-1}/\Delta$

and

$$\Delta = (1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2.$$

Note that Γ is a skew symmetric matrix. Further, (2.6) can be expressed as the matrix equation $\hat{\theta} = (A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}Y$, where $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$. Now, we will show that $Q(\theta)$, given by (2.5) is actually minimized when $\theta = \hat{\theta}$. For this, let us write

$$\begin{aligned} Q(\theta) &= (Y - A\hat{\theta} + A(\hat{\theta} - \theta))'\Sigma^{-1}(Y - A\hat{\theta} + A(\hat{\theta} - \theta)) \\ &= Q(\hat{\theta}) + 2(Y - A\hat{\theta})'\Sigma^{-1}A(\hat{\theta} - \theta) \\ &\qquad + (\hat{\theta} - \theta)'A'\Sigma^{-1}A(\hat{\theta} - \theta). \end{aligned} \tag{2.7}$$

Since $Y'\Sigma^{-1}A - \hat{\theta}'A'\Sigma^{-1}A = 0$, the middle term in (2.7) vanishes. The last term is always nonnegative, since Σ is a positive definite matrix. Hence, we can conclude that $Q(\theta)$ attains its minimum when $\theta = \hat{\theta}$, which means $\hat{\theta}$ is in fact the BLUE of θ.

Using (2.4) and (2.6), it is easily seen on simplification that

$$Var(\hat{\mu}) = \sigma^2\alpha'\Gamma\Sigma\Gamma'\alpha = \sigma^2\alpha'\Sigma^{-1}\alpha/\Delta, \tag{2.8}$$

$$Var(\hat{\sigma}) = \sigma^2 1'\Gamma\Sigma\Gamma'1 = \sigma^2 1'\Sigma^{-1}1/\Delta, \tag{2.9}$$

and

$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2\alpha'\Gamma\Sigma\Gamma'1 = -\sigma^2\alpha'\Sigma^{-1}1/\Delta. \tag{2.10}$$

These results can also be obtained by noting that

$$Cov(\hat{\theta}) = \sigma^2(A'\Sigma^{-1}A)^{-1}.$$

When the *pdf* of the standardized random variable X is symmetric around the origin, further simplification is possible. In that case,

$$(X_{1:n},\ldots,X_{i:n},\ldots,X_{n:n}) \underline{\underline{d}} (-X_{n:n},\ldots,-X_{n-i+1:n},\ldots,-X_{1:n}),$$

which can be represented as

$$X \underline{\underline{d}} -JX \tag{2.11}$$

where

$$J = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}$$

is a symmetric permutation matrix. Note that since $JJ' = I$, $J = J^{-1} = J'$. From (2.11) it then follows that $\alpha = -J\alpha$ and $\Sigma = J\Sigma J$. Hence,

$$\alpha'\Sigma^{-1}1 = \alpha'(J\Sigma J)^{-1}1 = \alpha'J\Sigma^{-1}J1 = -\alpha'\Sigma^{-1}1,$$

which implies that it must be zero. Thus, from (2.6) we obtain

$$\hat{\mu} = (1'\Sigma^{-1}Y)/(1'\Sigma^{-1}1), \qquad \hat{\sigma} = (\alpha'\Sigma^{-1}Y)/\alpha'\Sigma^{-1}\alpha. \tag{2.12}$$

Further,

$$Var(\hat{\mu}) = \sigma^2/1'\Sigma^{-1}1, \qquad Var(\hat{\sigma}) = \sigma^2/\alpha'\Sigma^{-1}\alpha, \tag{2.13}$$

and $Cov(\hat{\mu},\hat{\sigma}) = 0$.

So in the case of a symmetric population, the BLUEs of μ and σ are always uncorrelated.

REMARK 1: Even though the formulas for $\hat{\mu}$ and $\hat{\sigma}$ given in (2.12) are much simpler than those in the general case, they still depend on Σ^{-1}.

However, in the symmetric case the problem of finding the inverse of Σ can be reduced to inverting an associated matrix having only half the dimension of Σ. The technique exploits the fact that

$$\sigma_{i,j:n} = \sigma_{j,i:n} = \sigma_{n-j+1,n-i+1:n}$$

and the special form of $\hat{\mu}$ and $\hat{\sigma}$. See Balakrishnan *et al.*, (1992) for further details.

REMARK 2: A question of interest is whether $\bar{Y}_n$, which is a linear function of order statistics, can be the BLUE of the location parameter μ. For a symmetric parent this is possible only when $1'\Sigma^{-1} = 1'$ or $\Sigma 1 = 1$. In other words, when $\sum_{j=1}^{n} \sigma_{i,j:n} = 1$ for all i. This holds because, for the standard normal distribution, the sum of the elements in a row or column of the product-moment matrix or the covariance matrix of standard normal order statistics is 1 for any sample size n. Bondesson (1976) has shown that $\bar{Y}_n$ is the BLUE of μ for all n *if* and only *if* F_0 is either normal or a gamma translated to have mean 0. In all other cases $\bar{Y}_n$ is less efficient than $\hat{\mu}$.

REMARK 3: In the symmetric case, we have noted that $\hat{\mu}$ and $\hat{\sigma}$ are uncorrelated. $Cov(\hat{\mu}, \hat{\sigma})$ is zero *if* and only *if* $\alpha'\Sigma^{-1}1 = 0$. In that case, $\hat{\mu}$ and $\hat{\sigma}$ are given by (2.12), as in the symmetric case. Now for a gamma distribution translated to have mean 0, $\mu'1 = 0$ and since $\bar{Y}_n = 0$ is the BLUE of μ, $\Sigma^{-1}1 = 1$. Thus we would have $\alpha'\Sigma^{-1}1 = \alpha'1 = 0$, indicating the fact that $\hat{\mu}, \hat{\sigma}$ are correlated when F_0 is a gamma distribution translated to have zero mean.

REMARK 4: The discussion so far has concentrated on data consisting of the full set of order statistics. If the observed data consists of a fixed subset of order statistics to be labeled Y, the general formulae given in (2.6) and (2.8) to (2.10) continue to hold. This means we can use them to obtain the BLUEs and their moments when we have a Type II censored sample. The formula for the symmetric case hold whenever (2.11) is satisfied. This occurs, for example, when we have a Type II censored sample from a symmetric population where the censoring is also symmetric. Finally, note that formulas developed here do not yield the BLUEs when the sample is Type I censored.

3. BLUE OF THE PARAMETERS OF EXPONENTIAL DISTRIBUTION

3.1. ONE-PARAMETER EXPONENTIAL DISTRIBUTION

(Balakrishnan and Rao, 1997)

Let $X_{1:n}, X_{2:n}, \ldots, X_{n-s:n}$ be a Type-II right censored sample from a one-parameter exponential distribution with density

$$f(x;\sigma) = \frac{1}{\sigma} e^{-x/\sigma}, x \geq 0, \sigma > 0. \tag{3.1}$$

Then it is well known that the spacings

$$Z_1 = n X_{1:n},\ Z_2 = (n-1)(X_{2:n} - X_{1:n}), \ldots, Z_n = X_{n:n} - X_{n-1:n} \tag{3.2}$$

are all *i.i.d.* random variables exponentially distributed as in (3.1); for example, see Arnold *et al.* (1992). As a consequence of this result one also gets

$$\alpha_j = \sum_{i=1}^{j} 1/(n-i+1),\ \Sigma_{j,j} = \Sigma_{j,k} = \sum_{i=1}^{j} 1/(n-i+1)^2 \text{ (for } k > j).$$

In order to derive the BLUE of σ directly, without appealing to the results quoted in Section 2, let us write

$$\hat{\sigma} = \sum_{i=1}^{n-s} a_i X_{i:n} \tag{3.3}$$

or equivalently

$$\hat{\sigma} = \sum_{i=1}^{n-s} a_i' Z_i. \tag{3.4}$$

It is then clear that

$$E(\hat{\sigma}) = \sigma \sum_{i=1}^{n-s} a_i' \text{ and } Var(\hat{\sigma}) = \sigma^2 \sum_{i=1}^{n-s} a_i'^2. \tag{3.5}$$

Because $\hat{\sigma}$ needs to be unbiased for σ, all we need to do is to minimize $\sum_{i=1}^{n-s} a_i'^2$ subject to the condition $\sum_{i=1}^{n-s} a_i' = 1$, which immediately yields $a_i' = 1/(n-s)$, $i = 1, 2, \ldots, n-s$, and hence for $i = 1, 2, \ldots, n-s-1$, $a_i = 1/(n-s)$, and $a_{n-s} = (s+1)/(n-s)$. Then we obtain the BLUE of σ to be [from (3.3)]

$$\hat{\sigma} = \frac{1}{n-s} \left\{ \sum_{i=1}^{n-s-1} X_{i:n} + (s+1) X_{n-s:n} \right\} \tag{3.6}$$

from (3.5) we also readily have

$$Var(\hat{\sigma}) = \sigma^2 / (n-s). \tag{3.7}$$

Further, from (3.4) we also obtain immediately the well-known result that $2(n-s)\hat{\sigma}/\sigma$ is distributed as central chi-square with $2(n-s)$ degrees of freedom.

Next, let $X_{r+1:n}, X_{r+2:n}, \ldots, X_{n-s:n}$ be a Type- II doubly censored sample available from a one-parameter exponential distribution in (3.1). Then in order to derive the BLUE of σ of the form

$$\hat{\sigma} = \sum_{i=r+1}^{n-s} a_i X_{i:n} . \tag{3.8}$$

Let us start with

$$\hat{\sigma} = a'_{r+1} X_{r+1:n} + \sum_{i=r+2}^{n-s} a_i Z_i . \tag{3.9}$$

Because X_{r+1} and $Z_i = (i = r+2, \ldots, n-s)$ are statistically independent, we readily have

$$E(\hat{\sigma}) = \sigma \left\{ a'_{r+1} \alpha_{r+1} + \sum_{i=r+2}^{n-s} a'_i \right\} \tag{3.10}$$

and

$$Var(\hat{\sigma}) = \sigma^2 \left\{ {a'_{r+1}}^2 \Sigma_{r+1,r+1} + \sum_{i=r+2}^{n-s} a_i'^2 \right\} \tag{3.11}$$

Now minimizing the variance in (3.11) subject to the condition that $E(\hat{\sigma})$ in (3.10) equals σ, we immediately obtain

$$a'_{r+1} = \frac{\alpha_{r+1} / \Sigma_{r+1,r+1}}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})}$$

and

$$a'_i = \frac{1}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})}, \qquad i = r+2, \ldots, n-s.$$

Because

$$a_{r+1} = \frac{(\alpha_{r+1} / \Sigma_{r+1,r+1}) - (n-r-1)}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})},$$

$$a_i = \frac{1}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})}, \qquad i = r+2, \ldots, n-s-1$$

and

$$a_{n-s} = \frac{s+1}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})}.$$

Thus we obtain the BLUE of σ in this case to be

$$\hat{\sigma} = \frac{1}{n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1})}$$
$$\times \left[\left\{ \frac{\alpha_{r+1}}{\Sigma_{r+1,r+1}} (n-r-1) \right\} X_{r+1:n} + \sum_{i=r+2}^{n-s-1} X_{i:n} + (s+1) X_{n-s:n} \right].$$
$$(3.12)$$

From (3.11) we have

$$Var(\hat{\sigma}) = \sigma^2 / \{ n-s-r-1+(\alpha_{r+1}^2 / \Sigma_{r+1,r+1}) \}. \qquad (3.13)$$

These agree with the results presented, for example, in Balakrishnan and Cohen (1991).

3.2. Two-parameter exponential distribution

(Balakrishnan and Rao, 1997)

Let $X_{1:n}, X_{2:n}, \ldots, X_{n-s:n}$ be a Type-II right censored sample from a two-parameter exponential distribution with density

$$f(x;\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \qquad x \geq \mu, \sigma > 0. \qquad (3.14)$$

Then the variables Z_i's in (3.2) are still independent, with $Z_2, \ldots, Z_n$ distributed exponentially as in (3.1) and Z_1 distributed exponentially as in (3.14) with μ replaced by $n\mu$.

Next, in order to derive the BLUEs of μ and σ of the form

$$\hat{\mu} = \sum_{i=1}^{n-s} a_i X_{i:n} \qquad \text{and} \qquad \hat{\sigma} = \sum_{i=1}^{n-s} b_i X_{i:n}. \qquad (3.15)$$

Let us start with

$$\hat{\mu} = \sum_{i=1}^{n-s} a_i' Z_i \qquad \text{and} \qquad \hat{\sigma} = \sum_{i=2}^{n-s} b_i' Z_i . \tag{3.16}$$

It is clear that

$$E(\hat{\mu}) = a_1' \, n \, \mu + \sigma \sum_{i=1}^{n-s} a_i' , \qquad E(\hat{\sigma}) = \sigma \sum_{i=2}^{n-s} b_i',$$

$$Var(\hat{\mu}) = \sigma^2 \sum_{i=1}^{n-s} a_i'^2 , \qquad Var(\hat{\sigma}) = \sigma^2 \sum_{i=2}^{n-s} b_i'^2 ,$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2 \sum_{i=2}^{n-s} a_i' b_i' . \tag{3.17}$$

Because $\hat{\mu}$ and $\hat{\sigma}$ are required to be unbiased with the restrictions

$$a_1' = \frac{1}{n}, \qquad \sum_{i=1}^{n-s} a_i' = 0 \qquad \text{and} \qquad \sum_{i=2}^{n-s} b_i' = 1 . \tag{3.18}$$

The estimators of μ and σ are obtained by minimizing $\sum_{i=1}^{n-s} a_i'^2 + \sum_{i=2}^{n-s} b_i'^2$ subject to the conditions in (3.18). Then

$$a_1' = \frac{1}{n}, \qquad a_i' = -\frac{1}{n(n-s-1)}, \qquad i = 2,3,\ldots,n-s,$$

and

$$b_i' = \frac{1}{n-s-1}, \qquad i = 2,3,\ldots,n-s.$$

This yield

$$a_1 = 1 + \frac{n-1}{n(n-s-1)},$$

$$a_i' = -\frac{1}{n(n-s-1)}, \qquad i = 2\,,3,\ldots,n-s-1,$$

$$a_{n-s} = -\frac{s+1}{n(n-s-1)},$$

and

$$b_1 = -\frac{n-1}{n-s-1},$$

$$b_i = \frac{1}{n-s-1}, \qquad i=2,3,\ldots,n-s-1,$$

$$b_{n-s} = -\frac{s+1}{n-s-1}.$$

From (3.15) the linear unbiased estimators of μ and σ are

$$\hat{\mu} = X_{1:n} - \frac{1}{n(n-s-1)} \times \left\{ \sum_{i=2}^{n-s-1} X_{i:n} + (s+1)X_{n-s:n} - (n-1)X_{1:n} \right\}$$

$$= X_{1:n} - (\hat{\sigma}/n), \tag{3.19}$$

and

$$\hat{\sigma} = \frac{1}{n-s-1} \times \left\{ \sum_{i=2}^{n-s-1} X_{i:n} + (s+1)X_{n-s:n} - (n-1)X_{1:n} \right\}. \tag{3.20}$$

Also from (3.17)

$$Var(\hat{\mu}) = \frac{\sigma^2}{n^2(n-s-1)}(n-s),$$

$$Var(\hat{\sigma}) = \frac{\sigma^2}{(n-s-1)}$$

and

$$Cov(\hat{\mu},\hat{\sigma}) = -\frac{\sigma^2}{n(n-s-1)}$$

which all agree with the expressions given by Balakrishnan and Cohen (1991). From (3.16) we have the result that $2(n-s-1)\hat{\sigma}/\sigma$ has a central chi-square distribution with $2(n-s-1)$ degrees of freedom. For obtaining "determinant-efficient" estimators of μ and σ we minimize

$$\left(\frac{1}{n^2}+\sum_{i=2}^{n-s}a_i'^2\right)\left(\sum_{i=2}^{n-s}b_i'^2\right)-\left(\sum_{i=2}^{n-s}a_i'b_i'\right)^2 \tag{3.21}$$

subject to the conditions

$$\sum_{i=2}^{n-s}a_i'=-\frac{1}{n} \qquad \text{and} \qquad \sum_{i=2}^{n-s}b_i'=1. \tag{3.22}$$

Taking λ_1 and λ_2 as the Lagrangian multipliers, and then differentiating the objective function with respect to a_i', we obtain

$$2a_i'\sum_{j=2}^{n-s}b_j'^2-2b_i'\sum_{j=2}^{n-s}a_j'b_j'-\lambda_1=0 \tag{3.23}$$

which, when added over $i=2,3,\ldots,n-s$ and then simplified using (3.22), yields

$$\lambda_1=\frac{1}{n-s-1}\left\{-\frac{2}{n}\sum_{j=2}^{n-s}b_j'^2-2\sum_{j=2}^{n-s}a_j'b_j'\right\}.$$

Substituting this in (3.23) gives

$$\left(a_i'+\frac{1}{n(n-s-1)}\right)\sum_{j=2}^{n-s}b_j'^2-\left(b_i'-\frac{1}{n(n-s-1)}\right)\times\sum_{j=2}^{n-s}a_j'b_j'=0. \tag{3.24}$$

A similar treatment with respect to b_i' yields

$$\left(b_i'-\frac{1}{n(n-s-1)}\right)\left(\frac{1}{n^2}+\sum_{j=2}^{n-s}a_j'^2\right)$$
$$-\left(a_i'+\frac{1}{n(n-s-1)}\right)\sum_{j=2}^{n-s}a_j'b_j'=0. \tag{3.25}$$

Solving (3.24) and (3.25) simultaneously we obtain

$$a_i' = -\frac{1}{n(n-s-1)} \quad \text{and} \quad b_i' = \frac{1}{(n-s-1)} \qquad \text{for } i = 2,3,\ldots,n-s.$$

This implies that the best linear unbiased estimators of μ and σ in (3.19) and (3.20) are also the "determinant-efficient" linear unbiased estimators.

As a matter of fact, one may consider more generally the BLUE of $\theta = \alpha\mu + \beta\sigma$ (for given values of α and β) by the form

$$\hat{\theta} = \sum_{i=1}^{n-s} c_i X_{i:n} = \sum_{i=1}^{n-s} c_i' Z_i. \tag{3.26}$$

The condition of unbiasedness in this case readily gives

$$c_1' = \frac{\alpha}{n} \qquad \text{and} \quad \sum_{i=1}^{n-s} c_i' = \beta. \tag{3.27}$$

So we need to minimize $\sum_{i=2}^{n-s} c_i'^2$ subject to the condition

$$\sum_{i=1}^{n-s} c_i' = \beta - (\alpha / n),$$

which yields

$$c_2' = \cdots = c_{n-s}' = \frac{n\beta - \alpha}{n(n-s-1)}. \tag{3.28}$$

From (3.26) we then obtain

$$\hat{\theta} = \alpha\left\{\frac{Z_1}{n} - \frac{1}{n(n-s-1)}\sum_{i=2}^{n-s} Z_i\right\} + \frac{\beta}{n-s-1}\sum_{i=2}^{n-s} Z_i .$$

$$= \alpha\hat{\mu} + \beta\hat{\sigma}. \tag{3.29}$$

Furthermore, we have

$$Var(\alpha\hat{\mu} + \beta\hat{\sigma}) \leq Var(\alpha\mu^* + \beta\sigma^*) \tag{3.30}$$

for any other unbiased estimates $(\hat{\mu}, \hat{\sigma})$, which is

$$(\alpha \quad \beta)[Cov(\hat{\mu},\hat{\sigma})]_{2\times 2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \le (\alpha \quad \beta)[Cov(\mu^*,\sigma^*)]_{2\times 2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{3.31}$$

The inequality in (3.31) implies the complete covariance matrix dominance of the BLUEs, namely,

$$[Cov(\hat{\mu},\hat{\sigma})]_{2\times 2} \le [Cov(\mu^*,\sigma^*)]_{2\times 2}. \tag{3.32}$$

The covariance matrix dominance result in (3.32) is a very general result from which the trace efficiency and determinant efficiency readily follow as special cases.

4. Blue Of The Parameters Of The Logistic Distribution

(*Gupta et al., 1967*)

The logistic distribution is described by the *pdf*

$$f(x;\mu,\sigma) = (1/\sigma')\exp\{-(x-\mu)/\sigma'\}/[1+\exp\{-(x-\mu)/\sigma'\}]^2,$$

$$-\infty < x < \infty, \quad -\infty < \mu < \infty \tag{4.1}$$

where $\sigma' = \dfrac{\sqrt{3}\sigma}{\pi} > 0.$

This distribution is symmetrical with mean μ and variance σ^2.

In this section, linear unbiased estimators with minimum variance based on ordered observations are constructed from the complete as well as the censored sample case. The censored sample case consists of the $n-n_1-n_2$ observations, where n_1 observations are missing in the beginning and n_2 observations are missing at the end.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the n ordered observations from the logistic distribution, with mean μ and variance σ^2 which has the *pdf* (4.1). We are interested in constructing estimators $\hat{\mu}$ and $\hat{\sigma}$ based on the

$n-n_1-n_2$ observations $X_{n_1+1} \le X_{n_1+2} \le \ldots \le X_{n-n_2}$, where n_1 observations are missing on the left and n_2 observations are missing on the right.

Let the required estimators be

$$\hat{\mu} = \sum_{i=n_1+1}^{n-n_2} a_{i,n} X_{i:n} \tag{4.2}$$

$$\hat{\sigma} = \sum_{i=n_1+1}^{n-n_2} b_{i,n} X_{i:n} . \tag{4.3}$$

The problem is to find the coefficients $a_{i,n}$ and $b_{i,n}$ such that, in the class of linear unbiased estimators, $\hat{\mu}$ and $\hat{\sigma}$ have minimum variance.

It follows from the condition of unbiasedness that

$$\sum a_{i,n} = 1 \quad \text{and} \quad \sum a_{i,n} \mu'_1(i,n) = 0 \tag{4.4}$$

$$\sum b_{i,n} = 0 \quad \text{and} \quad \sum b_{i,n} \mu'_1(i,n) = 1 \tag{4.5}$$

where $\mu'_1(i,n)$ is the first moment or cumulant of the i^{th} order statistic in a sample of size n from the logistic distribution $L(0,1)$, i.e., the distribution with mean zero and variance unity. It should be pointed out that (4.4) and (4.5) are necessary conditions.

The estimators $\hat{\mu}$ and $\hat{\sigma}$ can be written, using the matrix notation and following the generalized least-squares theory (Lloyd, 1952) in the following form

$$\hat{\mu} = -\alpha' \Gamma \mathrm{X}, \tag{4.6}$$

$$\hat{\sigma} = 1' \Gamma \mathrm{X}, \tag{4.7}$$

where

$$\mathrm{X} = (X_{n_1+1:n}, \ldots, X_{n-n_2:n})$$

$\alpha' = (\mu_1'(n_1+1, n), \ldots, \mu_1'(n-n_2, n))$

$1' = (1, \ldots, 1)$, $1'$ is a vector with $n - n_1 - n_2$ components

$\Gamma = (\Sigma^{-1}(1\alpha' - \alpha' 1)\Sigma^{-1}) / \Delta$

$\Delta = (1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2$

and where Σ is the variance-covariance matrix of $n - n_1 - n_2$ appropriate order statistics.

The special case $n_1 = 0$, $n_2 = 0$ is of importance, since this represents the complete sample case.

The variances and covariances of $\hat{\mu}$ and $\hat{\sigma}$ can be expressed as

$$Var(\hat{\mu}) = (\alpha'\Sigma^{-1}\alpha)\sigma^2 / \Delta \tag{4.8}$$

$$Var(\hat{\sigma}) = (1'\Sigma^{-1}1)\sigma^2 / \Delta \tag{4.9}$$

$$Cov(\hat{\mu}, \hat{\sigma}) = -(1'\Sigma^{-1}\alpha)\sigma^2 / \Delta. \tag{4.10}$$

When $n_1 = n_2$, the symmetry of the distribution implies that $Cov(\hat{\mu}, \hat{\sigma}) = 0$.

Thus, for constructing $\hat{\mu}$ and $\hat{\sigma}$, one needs the expected values of $X_{r:n}, X_{r:n}^2$ and $(X_{r:n} X_{s:n})$. The first two of these are available in closed form and also numerically in the papers by Plackett (1958), Birnbaum and Dudman (1963) ,Gupta and Shah (1965) and Khan *et al*(1983 a). In a paper by Shah (1965) and Khan *et al*(1983 b), the covariances of the order statistics $X_{r:n}$ and $X_{s:n}$ have been expressed in a closed form in terms of digamma and trigamma functions where numerical values of these covariances are also given for $n = 2(1)10$. In the paper by Gupta *et al.*, 1967) variances and covariances for $n = 10(5)25$ and the coefficients of BLUEs of the mean and standard deviation for both complete and censored samples for $n = 2, 5, 10, 15, 20$ and 25 have been tabulated.

4.1: Relative efficiencies of the estimators for the uncensored sample cases

Now the relative efficiencies of the estimator $\hat{\mu}$ and $\hat{\sigma}$ will be discussed for the uncensored sample cases. In developing various formulae, we will follow the general notation as described in Section 2 for the special case since the uncensored case is simply the one where $n_1 = 0$ and $n_2 = 0$.

The relative efficiency (Rel. Eff.) is defined in terms of the reciprocal of the ratio of the variances of these estimators to the lower bound on these variances as obtained by the Cramer-Rao inequality. These bounds are

$$Var(\hat{\mu}) \geq (9/\pi^2)(\sigma^2/n) \tag{4.11}$$

$$Var(\hat{\sigma}) \geq (9/(3+\pi^2))(\sigma^2/n). \tag{4.12}$$

Hence

$$\text{Rel. Eff. of } \hat{\mu} = (9/n\pi^2)(\Delta/(\alpha'\Sigma^{-1}\alpha)) \tag{4.13}$$

$$\text{Rel. Eff. of } \hat{\sigma} = 9/(n(3+\pi^2))(\Delta/(1'\Sigma^{-1}1)). \tag{4.14}$$

We will also study the efficiencies of these estimators as compared with the moment estimators. The moment estimator $\bar{X}$ for μ has the variance σ^2/n and hence

$$\text{Rel. Eff. of } \bar{X} = \frac{Var(\hat{\mu})}{Var(\bar{X})} = n(\alpha'\Sigma^{-1}\alpha)/\Delta \tag{4.15}$$

The moment estimator for σ is $\sqrt{m_2}$ where m_2 is the second sample moment based on n observations. Now

$$E(\sqrt{m_2}) = \sigma + O(1/n), \quad Var(\sqrt{m_2}) = (4/5n) + O(1/n^2) \tag{4.16}$$

The $Var(\sqrt{m_2})$ can be better approximated by

$$Var(\sqrt{m_2}) \approx (n-1)(31n-21)/(20n^2). \tag{4.17}$$

[The formulae (4.16) and (4.17) are available in Cramer (1951), Kendall and Stuart (1963)].

Hence,

$$\begin{aligned}\text{Rel.Eff. of } \sqrt{m_2} &\approx Var(\hat{\sigma})/Var(\sqrt{m_2}) \\ &= ((20n^3)/(n-1)(31n-21))((1'V^{-1}1)/\Delta). \qquad (4.18)\end{aligned}$$

The relative efficiency of the moment estimators decreases as n increases. It should be pointed out that the variance of $\hat{\mu}$ has a value strictly smaller than σ^2/n, since it has been obtained by constructing a linear compound of the observations with minimal variance and, therefore, it cannot exceed the variance of the particular linear compound which is the sample mean. The covariances of order statistics were computed using the result in Shah (1965), which is,

$$\begin{aligned}E(X_{i:n}\, X_{j:n}) = E(X_{i:n}^2) &+ \sum_{k=i}^{i-1}\sum_{t=1}^{k-1}(-1)^{k+i}\binom{k-1}{i-1}\binom{n}{k} \\ &\times\binom{j-k+t-1}{t}B(t,n-k+1)\; E(X_{j+t:n+t-k}) \\ &+\binom{n}{i}\sum_{k=0}^{j-i-1}(-1)^k\binom{n-i}{k}(1/(i+k))\{-\psi^{(1)}(n-j) \\ &+(\psi^{(0)}(n-j)-\psi^{(0)}(n-i-k)) \\ &\times(\psi^{(0)}(j-i-k-1)\psi^{(0)}(n-j))\}) \qquad (4.19)\end{aligned}$$

where $B(p,q)$ is the beta function and

$$\psi^{(r-1)}(x) = \frac{d^r \log\Gamma(1+x)}{dx^r}.$$

By using the known result that

$$Cov(X_{i:n}, X_{j:n}) = Cov(X_{n-j+1}, X_{n-i+1}), \qquad i < j \tag{4.20}$$

checks were made on the accuracy of the covariances.

Remark: It is interesting to note that if the sample size is odd and all the observations are censored except the middle observation, then the middle observation will have all the weight in estimating μ. And if the sample size is even and all the observations are censored except the middle two, then the weight of each observation will be one-half in estimating μ. In such a situation, correlation between μ and σ is always zero. The above remark is true for any symmetrical distribution. One can notice these facts for the normal distribution also. Thus, the weights for μ, are the same for all symmetrical distributions. Inference based on this fact will be misleading without the prior knowledge of their parent population.

Other features about the relative efficiencies of $\hat{\mu}$ and $\hat{\sigma}$ for censored samples as noted by Sarhan and Greenberg (1962) for normal distribution are also true here for logistic distribution.

5. Blue Of The Parameters Of The Generalized Pareto Distribution (*Mahmoud et al., 2005*)

The *pdf* of the generalized Pareto distribution (GPD) is given by

$$f(x) = \begin{cases} \frac{1}{\sigma}\left\{1+\beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-(1+1/\beta)} & , x \geq \mu, for\ \beta > 0, \\ & \mu < x < \mu - \sigma/\beta\ for\ \beta < 0, \\ \frac{1}{\sigma}e^{-(x-\mu)/\sigma} & , x \geq \mu, \quad for\ \beta = 0, \\ 0 & , \text{otherwise} \end{cases} \tag{5.1}$$

while the standard form of the GPD is given by

$$f(z)=\begin{cases}\{1+\beta z\}^{-(1+1/\beta)} & ,z\geq 0,\text{ for }\beta>0,\\ & 0<z<-1/\beta\text{ for }\beta<0,\\ e^{-z} & ,z\geq 0,\text{ for }\beta=0,\\ 0 & ,\text{otherwise}\end{cases} \tag{5.2}$$

REMARKS

Let $X_{1:n},X_{2:n},\ldots,X_{n:n}$ denote the available order statistics from the GPD in (5.1), and let $Z_{i:n}=(X_{i:n}-\mu)/\sigma, i=1,2,\ldots,n,$ be the corresponding order statistics from the standard GPD in (5.2). Let us denote $E(Z_{i:n})$ by $\mu_{i:n}$,$Var(Z_{i:n})$ by $\sigma_{i:n}$, and $Cov(Z_{i:n},Z_{j:n})$ by $\sigma_{i,j:n}$; furthermore, let

$$\mathrm{X}'=(X_{1:n},X_{2:n},\ldots,X_{n:n}),$$

$$\alpha'=(\mu_{1:n},\mu_{2:n},\ldots,\mu_{n:n}),$$

$$1'=\underbrace{(1,1,\ldots,1)}_{n},\text{ and }\sum=(\sigma_{i,j:n}),\quad 1\leq i,j\leq n.$$

Then, the BLUEs of μ and σ are given by (Balakrishnan and Cohen, 1991)

$$\hat{\mu}=\left\{\frac{\alpha'\Sigma^{-1}\alpha 1'\Sigma^{-1}-\alpha'\Sigma^{-1}1\alpha'\Sigma^{-1}}{(\alpha'\Sigma^{-1}\alpha)(1'\Sigma^{-1}1)-(\alpha'\Sigma^{-1}1)^2}\right\}\mathrm{X}=\sum_{i=1}^{n}A_i\,X_{i:n}, \tag{5.3}$$

$$\hat{\sigma}=\left\{\frac{1'\Sigma^{-1}1\alpha'\Sigma^{-1}-1'\Sigma^{-1}\alpha 1'\Sigma^{-1}}{(\alpha'\Sigma^{-1}\alpha)(1'\Sigma^{-1}1)-(\alpha'\Sigma^{-1}1)^2}\right\}\mathrm{X}=\sum_{i=1}^{n}B_i\,X_{i:n}. \tag{5.4}$$

Furthermore, the variances and covariance of these BLUEs are given by (Balakrishnan and Cohen, 1991)

$$Var(\hat{\mu}) = \sigma^2 \left\{ \frac{\alpha' \Sigma^{-1} \alpha}{(\alpha' \Sigma^{-1} \alpha)(1' \Sigma^{-1} 1) - (\alpha' \Sigma^{-1} 1)^2} \right\} = \sigma^2 V_1 , \qquad (5.5)$$

$$Var(\hat{\sigma}) = \sigma^2 \left\{ \frac{1' \Sigma^{-1} 1}{(\alpha' \Sigma^{-1} \alpha)(1' \Sigma^{-1} 1) - (\alpha' \Sigma^{-1} 1)^2} \right\} = \sigma^2 V_2 , \qquad (5.6)$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2 \left\{ \frac{-\alpha' \Sigma^{-1} 1}{(\alpha' \Sigma^{-1} \alpha)(1' \Sigma^{-1} 1) - (\alpha' \Sigma^{-1} 1)^2} \right\} = \sigma^2 V_3 , \qquad (5.7)$$

for details, see David (1981), Balakrishnan and Cohen (1991), and Arnold *et al.* (1992). Exact explicit expressions can be obtained through (5.3) to (5.7) by using the fact that the covariance matrix of the standardized generalized Pareto order statistics ($\sigma_{i,j:n}$) is of the form $(a_i b_j)$.

The single and product moments of order statistics from the standard form of the GPD are given as

$$\mu_{i:n} = \frac{1}{\beta} \left[\frac{\Gamma(n+1)\Gamma(n-i-\beta+1)}{\Gamma(n-i+1)\Gamma(n-\beta+1)} - 1 \right] \qquad (5.8)$$

$$\mu_{i:n}^{(2)} = \frac{1}{\beta^2} \left[1 - \frac{2\Gamma(n-i-\beta+1)\Gamma(n+1)}{\Gamma(n-i+1)\Gamma(n-\beta+1)} + \frac{\Gamma(n+1)\Gamma(n-i-2\beta+1)}{\Gamma(n-i+1)\Gamma(n-2\beta+1)} \right], \qquad (5.9)$$

$$\mu_{i,j:n} = \frac{1}{\beta^2} \left[1 - \frac{\Gamma(n+1)\Gamma(n-i-\beta+1)}{\Gamma(n-i+1)\Gamma(n-\beta+1)} - \frac{\Gamma(n+1)\Gamma(n-j-\beta+1)}{\Gamma(n-j+1)\Gamma(n-\beta+1)} \right.$$
$$\left. + \frac{\Gamma(n+1)\Gamma(n-j-\beta+1)\Gamma(n-i-2\beta+1)}{\Gamma(n-j+1)\Gamma(n-i-\beta+1)\Gamma(n-2\beta+1)} \right], \qquad (5.10)$$

then, the variance-covariance matrix of order statistics from the standardized generalized Pareto can be written as $((\sigma_{i,j:n})) = ((a_i b_j))$ where

$$a_i = \frac{1}{\beta^2}\left[\frac{\Gamma(n-2\beta-i+1)}{\Gamma(n-i-\beta+1)\Gamma(n-2\beta+1)} - \frac{\Gamma(n+1)\,\Gamma(n-i-\beta+1)}{\Gamma(n-i+1)\,(\Gamma(n-\beta+1))^2}\right] \tag{5.11}$$

and

$$b_j = \frac{\Gamma(n+1)\Gamma(n-j+1-\beta)}{\Gamma(n-j+1)}. \tag{5.12}$$

We are therefore able to invert the covariance matrix $((\sigma_{i,j:n}))$ and obtain explicit expressions for the BLUEs $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ respectively, their variances $Var(\hat{\mu})$ and $Var(\hat{\sigma})$ and their covariance $Cov(\hat{\mu},\hat{\sigma})$, as follows

$$\hat{\mu} = \sum_{j=1}^{n} A_j\, X_{j:n} = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{C_{i,j}(\gamma_1-\gamma_2\mu_{i:n})}{\gamma_1\zeta-\gamma_2^2}\right)X_{j:n}, \tag{5.13}$$

$$\hat{\sigma} = \sum_{j=1}^{n} B_j\, X_{j:n} = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{C_{i,j}(\zeta_1\mu_{i:n}-\gamma_2)}{\gamma_1\zeta-\gamma_2^2}\right)X_{j:n}, \tag{5.14}$$

$$Var(\hat{\mu}) = \frac{\sigma^2\gamma_1}{\gamma_1\zeta_1-\gamma_2^2} \quad Var(\hat{\sigma}) = \frac{\sigma^2\zeta_1}{\gamma_1\zeta_1-\gamma_2^2}, \tag{5.15}$$

where

$$\gamma_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{i,j}\mu_{i:n}\,\mu_{j:n}, \quad \gamma_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{i,j}\mu_{i:n} \text{ and } \zeta_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{i,j} \tag{5.16}$$

and

$$C_{i,j}=\begin{cases}\dfrac{-(n-i)(n-i-\beta)\Gamma(n-i)\Gamma(n-2\beta+1)}{\Gamma(n+1)\Gamma(n-2\beta-i)}, & j=i+1, i=1,\dots,n-1,\\[2ex] \dfrac{\Gamma(n-i+1)\Gamma(n-2\beta+1)}{\Gamma(n+1)\Gamma(n-2\beta-i+1)} & \\ \quad\times\left[(n-i-\beta)(2n-2i-\beta-1)\right. & \\ \quad\left.+(n-i+1)(1-\beta)\right], & i=j=2\ to\ (n-1),\\ \dfrac{n-2\beta}{n\beta^2}[(n-\beta)^2(n-\beta-1)^2 & \\ \quad -n(n-1)(n-2\beta)(n-2\beta-1)], & i=j=1,\\ \dfrac{(1-\beta^2)\Gamma(n-2\beta+1)}{\Gamma(1-2\beta)\Gamma(n+1)}, & i=j=n,\\ 0, & \text{otherwise}.\end{cases} \tag{5.17}$$

From (5.13) and (5.14), it is easy to show that

$$\sum_{i=1}^{n}A_i=\sum_{i=1}^{n}\frac{C_{i,j}(\gamma_1-\gamma_2\mu_{i:n})}{\gamma_1\zeta_1-\gamma_2^2}=1$$

and

$$\sum_{i=1}^{n}B_i=\sum_{i=1}^{n}\frac{C_{i,j}(\zeta_1\mu_{i:n}-\gamma_2)}{\gamma_1\zeta_1-\gamma_2^2}=0$$

where A_i and B_i are the coefficients of the BLUEs.

By making use of the exact explicit expressions (5.13) and (5.14), the coefficients A_i and B_i , $i=1,\dots,n$ were calculated for sample sizes $n=5(5)20(10)50$ and $\beta=-0.3,-0.1,0.1,0.3$ (Mahmoud *et al.*,2005).

6. BLUE Of The Parameters Of Uniform Distribution

(*Ahsanullah and Nevzorov, 2005*)

Suppose $X_{1:n},X_{2:n},\dots,X_{n:n}$ are n *i.i.d.* uniform random variables with *pdf* $f(x)$ given as

$$f(x)=\frac{1}{\sigma} \quad ,\mu-\sigma/2\leq x\leq\mu+\sigma/2,\ -\infty<\mu<\infty,\sigma>0 \tag{6.1}$$

$$=0 \quad \text{, otherwise.}$$

We have the moments of order statistics as

$$E(X_{r:n})=\mu+\sigma\left(\frac{r}{n+1}-\frac{1}{2}\right), \tag{6.2}$$

$$Var(X_{r:n})=\frac{r(n-r+1)}{(n+1)^2(n+2)}\sigma^2, r=1,2,\ldots.n, \tag{6.3}$$

$$Cov(X_{r:n},X_{s:n})=\frac{r(n-s+1)}{(n+1)^2(n+2)}\sigma^2, 1\leq r\leq s\leq n. \tag{6.4}$$

The inverse of variance-covariance matrix $\Sigma^{-1}(=\sigma^{ij})$ can be expressed as

$$\sigma^{ij}=\begin{cases} 2(n+1)(n+2), & i=j=1,2,\ldots,n \\ -(n+1)(n+2), & j=i+1, i=1,2,\ldots,n-1 \\ 0, & |i-j|>1 \end{cases} \tag{6.5}$$

It can be easily verified that

$$1'\Sigma^{-1}=((n+1)(n+2),0,0,\ldots,0,(n+1)(n+2)),$$

$$1'\Sigma^{-1}1=2(n+1)(n+2),$$

$$1'\Sigma^{-1}\alpha=0,$$

$$\alpha'\Sigma^{-1}=\left(-\frac{(n+1)(n+2)}{2},0,0,\ldots,0,\frac{(n+1)(n+2)}{2}\right)$$

$$\alpha'\Sigma^{-1}\alpha=\frac{(n+1)(n+2)}{2}.$$

Thus the BLUEs $\hat{\mu},\hat{\sigma}$ of μ and σ are

$$\hat{\mu}=\frac{1'\Sigma^{-1}X}{1'\Sigma^{-1}1}=\frac{X_{1:n}+X_{n:n}}{2} \tag{6.6}$$

and

$$\hat{\sigma}=\frac{\alpha'\Sigma^{-1}X}{\alpha'\Sigma^{-1}\alpha}=\frac{(n+1)(X_{n:n}-X_{1:n})}{n-1}. \tag{6.7}$$

The corresponding covariance of the estimators is zero and their variances are

$$Var(\hat{\mu})=\frac{\sigma^2}{1'\Sigma^{-1}1}=\frac{\sigma^2}{2(n+1)(n+2)} \tag{6.8}$$

$$Var(\hat{\sigma})=\frac{\sigma^2}{\alpha'\Sigma^{-1}\alpha}=\frac{2\sigma^2}{(n-1)(n+2)}. \tag{6.9}$$

In case of censored sample, where n_1 observations are missing on the left and n_2 observations are missing on the right, the inverse Σ^{-1} of the corresponding matrix is

$$\Sigma^{-1}=(n+1)(n+2)\begin{pmatrix} \frac{n_1+2}{n_1+1} & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ . & . & . & . & . & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{n_2+2}{n_2+1} \end{pmatrix} \tag{6.10}$$

The BLUEs $\hat{\mu},\hat{\sigma}$ of μ and σ are respectively

$$\hat{\mu}=\frac{1}{\Delta}(\alpha'\Sigma^{-1}\alpha 1'\Sigma^{-1}-\alpha'\Sigma^{-1}1\alpha'\Sigma^{-1})X$$

$$\hat{\sigma}=\frac{1}{\Delta}(1'\Sigma^{-1}1\alpha'\Sigma^{-1}-1'\Sigma^{-1}\alpha 1'\Sigma^{-1})X,$$

where

$$\Delta=(\alpha'\Sigma^{-1}\alpha)(1'\Sigma^{-1}1)-(\alpha'\Sigma^{-1}1)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 (\alpha' \Sigma^{-1} \alpha) / \Delta$$

$$Var(\hat{\sigma}) = \sigma^2 (1' \Sigma^{-1} 1) / \Delta$$

$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 (1' \Sigma^{-1} \alpha) / \Delta.$$

On simplification we get

$$\hat{\mu} = \frac{(n - 2n_2 - 1) X_{n_1+1:n} + (n - 2n_1 - 1) X_{n-n_2:n}}{2(n - n_1 - n_2 - 1)} \qquad (6.11)$$

and

$$\hat{\sigma} = \frac{(n+1)(X_{n-n_2:n} - X_{n_1+1:n})}{(n - n_1 - n_2 - 1)}. \qquad (6.12)$$

The variance and the covariance of the estimators are

$$Var(\hat{\mu}) = \frac{(n_1+1)(n - 2n_2 - 1) + (n_2+1)(n - 2n_1 - 1)}{4(n+1)(n+2)(n - n_1 - n_2 - 1)} \sigma^2, \qquad (6.13)$$

$$Var(\hat{\sigma}) = \frac{n_1 + n_2 + 2}{(n+2)(n - n_1 - n_2 - 1)} \sigma^2, \qquad (6.14)$$

$$Cov(\hat{\mu}, \hat{\sigma}) = \frac{1}{2(n+1)(n+2)} [(n_2+1)(n - 2n_1 - 1)(n - n_2)$$

$$- (n - 2n_2 - 1)(n_1 + 1) - 2(n_2 - n_1)(n_1+1)(n_2+1)].$$

$$(6.15)$$

If $n_1 = n_2 = r$, then

$$\hat{\mu} = \frac{X_{r+1:n} + X_{n-r:n}}{2} \hat{\sigma}, \qquad (6.16)$$

$$\hat{\sigma} = \frac{(n+1)(X_{n-r:n} - X_{r+1:n})}{(n - 2r - 1)} \qquad (6.17)$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = 0.$$

CHAPTER IV

BEST LINEAR UNBIASED ESTIMATION FOR RECORD VALUES

1. INTRODUCTION

In this chapter, BLUEs of scale and location parameters for exponential, generalized Pareto, power function, Weibull and Rayleigh distributions have been considered based on record values. The variance and covariance of the parameters are also discussed here.

2. BEST LINEAR UNBIASED ESTIMATORS FOR GENERAL LOCATION-SCALE FAMILY DISTRIBUTION *(Balakrishnan and Cohen, 1991)*

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the upper record values observed from a general location-scale family of distribution with *cdf* of the form $F\left(\frac{x-\mu}{\sigma}\right)$ and *pdf* of the form $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$, then the BLUEs of μ and σ are

$$\hat{\mu} = -\alpha' \Sigma^{-1}(1\alpha' - \alpha 1')\Sigma^{-1} X / \Delta \tag{2.1}$$

$$\hat{\sigma} = 1'\Sigma^{-1}(1\alpha' - \alpha 1')\Sigma^{-1} X / \Delta, \tag{2.2}$$

where

$$\Delta = (1'\Sigma^{-1} 1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2$$

and $X' = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$, α denotes the column vector of expected value of the upper record values for the given distribution, Σ denotes the variance-covariance matrix of the upper record values and 1 is a column vector with all its entries as 1. The variance and covariance of these BLUEs are given by

$$Var(\hat{\mu}) = \sigma^2 \alpha' \Sigma^{-1} \alpha / \Delta$$

$$Var(\hat{\sigma}) = \sigma^2 1' \Sigma^{-1} 1 / \Delta$$

$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 1' \Sigma^{-1} \alpha / \Delta.$$

2.1: WEIBULL DISTRIBUTION *(Ahsanullah, 2004)*

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from an *i.i.d.* sequence from Weibull distribution with *pdf* (assuming the shape parameter γ as known)

$$f(x) = \frac{(x-\mu)^{\gamma-1}}{\sigma^\gamma} e^{-\frac{(x-\mu)^\gamma}{\gamma\sigma^\gamma}}, \quad \mu < x < \infty, \sigma > 0. \qquad (2.3)$$

Let $X' = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$, then

$$Var(X) = \sigma^2 \Sigma, \quad \Sigma = (\sigma_{i,j}), \quad \sigma_{i,j} = a_i b_j, \quad 1 \le i \le j \le n$$

and $\sigma_{i,j} = \sigma_{j,i}$.

We can express the inverse of Σ as $\Sigma^{-1}(=\sigma^{ij})$

$$\sigma^{i+1,i} = \sigma^{i,i+1} = \frac{-1}{a_{i+1}b_i - a_i b_{i+1}} = -\gamma^{-2/\gamma} i\gamma(i\gamma+1)\frac{\Gamma(i)}{\Gamma(i+2/\gamma)},$$

$$i = i = 1, 2, \ldots, n-1 \qquad (2.4)$$

$$\sigma^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1, 2, \ldots, n \qquad (2.5)$$

$$\sigma^{i,j} = 0 \text{ for } |i-j| > 1$$

where $a_0 = 0 = b_{n+1}$ and $b_0 = 0 = a_{n+1}$.

On simplification, we obtain

$$\sigma^{i,i} = -\gamma^{-2/\gamma}\frac{\Gamma(i)}{\Gamma(i+2/\gamma)}[\gamma^2(2i^2-2i+1)+\gamma(4i-2)+1],$$

$$i=1,2,\ldots,n-1 \tag{2.6}$$

and

$$\sigma^{n,n} = -\gamma^{-2/\gamma}\frac{\Gamma(n)b_{n-1}}{\Gamma(n+2/\gamma)b_n}[(n\gamma-\gamma+1)(n\gamma-\gamma+2)]. \tag{2.7}$$

The BLUEs $\hat{\mu},\hat{\sigma}$ of μ and σ are respectively

$$\hat{\mu} = -\alpha'\Sigma^{-1}(1\alpha' - \alpha 1')\Sigma^{-1}X/\Delta$$

$$\hat{\sigma} = -1'\Sigma^{-1}(1\alpha' - \alpha 1')\Sigma^{-1}X/\Delta,$$

where

$$\Delta = (1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2.$$

On simplification, we obtain the BLUEs $\hat{\mu},\hat{\sigma}$ of μ and σ as

$$\hat{\mu} = \sum_{i=1}^{n} c_i X_{U(i)}, \tag{2.8}$$

and $$\hat{\sigma} = \sum_{i=1}^{n} d_i X_{U(i)} \tag{2.9}$$

and the corresponding variances and covariances are

$$Var(\hat{\mu}) = \sigma^2\frac{\alpha_n b_n}{D} \tag{2.10}$$

$$Var(\hat{\sigma}) = \sigma^2\frac{b_n^2}{D}T \tag{2.11}$$

$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2\frac{b_n}{D} \tag{2.12}$$

where

$$c_1 = \frac{\alpha_n b_n}{D} \frac{(\gamma+1)\gamma^{-2/\gamma}}{\Gamma(1+1/\gamma)}, c_i = \frac{\alpha_n b_n}{D} \frac{(\gamma-1)\gamma^{-2/\gamma}\Gamma(i)}{\Gamma(i+2/\gamma)},$$

$$i = 2,3,\ldots,n-1,$$

$$c_n = 1\frac{\alpha_n b_n}{D}\gamma^{-2/\gamma}\left[\frac{(\gamma+1)}{\Gamma(1+2/\gamma)} + (\gamma-1)\sum_{i=2}^{n-1}\frac{\Gamma(i)}{\Gamma(i+2/\gamma)}\right],$$

$$d_1 = -\frac{b_n(\gamma+1)}{D}\gamma^{-2/\gamma}$$

$$d_i = -\frac{b_n(\gamma-1)}{D}\gamma^{-2/\gamma}\frac{\Gamma(i)}{\Gamma(i+2/\gamma)}, \qquad i = 2,3,\ldots,n-1,$$

$$d_n = \frac{b_n}{D}\gamma^{-2/\gamma}\left[\frac{(\gamma+1)}{\Gamma(1+2/\gamma)} + (\gamma-1)\sum_{i=2}^{n-1}\frac{\Gamma(i)}{\Gamma(i+2/\gamma)}\right],$$

$$D = \alpha_n b_n T - 1,$$

$$T = \gamma^{-2/\gamma}\left[\frac{(\gamma+1)}{\Gamma(1+2/\gamma)} + (\gamma-1)\sum_{i=2}^{n-1}\frac{\Gamma(i)}{\Gamma(i+2/\gamma)} + \frac{(n+1)}{\Gamma(n+2/\gamma)}(n\gamma-\gamma+2)\left(\frac{b_{n-1}}{b_n}-1\right)\right]$$

and

$$\alpha_n = \gamma^{1/\gamma}\frac{\Gamma(n+1/\gamma)}{\Gamma(n)}, \qquad b_n = \gamma^{1/\gamma}\frac{\Gamma(n+2/\gamma)\Gamma(n+1/\gamma)}{\Gamma(n+1/\gamma)\Gamma(n)}.$$

2.2: RAYLEIGH DISTRIBUTION (*Ahsanullah, 2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from an *i.i.d.* sequence from Rayleigh distribution with *pdf*

$$f(x) = \frac{x-\mu}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \mu < x < \infty, \sigma > 0. \tag{2.13}$$

Let $X' = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$, then

$$E(X) = \mu 1 + \sigma \alpha$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$\alpha_i = \sqrt{2}\,\frac{\Gamma(i+1/2)}{\Gamma(i)}, \qquad i = 1, 2, \ldots, n. \tag{2.14}$$

$$Var(X) = \sigma^2 \Sigma, \quad \Sigma = (\sigma_{ij}), \sigma_{ij} = a_i b_j, 1 \le i \le j \le n$$

and $\sigma_{i,j} = \sigma_{j,i}$.

We can express the inverse of Σ as $\Sigma^{-1} (= \sigma^{i,j})$

$$\sigma^{i+1,i} = \sigma^{i,i+1} = \frac{-1}{a_{i+1} b_i - a_i b_{i+1}} = -(2i+1), \quad i = 1, 2, \ldots, n-1$$

$$\sigma^{i,i} = \frac{a_{i+1} b_{i-1} - a_{i-1} b_{i+1}}{(a_i b_{i-1} - a_{i-1} b_i)(a_{i+1} b_i - a_i b_{i+1})}, \qquad i = 1, 2, \ldots, n$$

$$\sigma^{i,j} = 0 \ \text{ for } \ |i-j| > 1$$

where $a_0 = 0 = b_{n+1}$ and $b_0 = 0 = a_{n+1}$.

On simplification, we obtain

$$\sigma^{i,i} = \frac{8i^2+1}{2i}, \qquad i = 1, 2, \ldots, n-1,$$

and

$$\sigma^{n,n} = (2n-1)\frac{b_{n-1}}{b_n}.$$

The BLUEs $\hat{\mu}, \hat{\sigma}$ of μ and σ are respectively

$$\hat{\mu} = -\alpha' \Sigma^{-1} (1\alpha' - \alpha 1') \Sigma^{-1} X / \Delta$$

$$\hat{\sigma} = -1' \Sigma^{-1} (1\alpha' - \alpha 1') \Sigma^{-1} X / \Delta,$$

where

$$\Delta = (1' \Sigma^{-1} 1)(\alpha' \Sigma^{-1} \alpha) - (1' \Sigma^{-1} \alpha)^2.$$

On simplification, we obtain the BLUEs $\hat{\mu},\hat{\sigma}$ of μ and σ as

$$\hat{\mu}=\sum_{i=1}^{n} c_i X_{U(i)}, \qquad \text{and} \qquad \hat{\sigma}=\sum_{i=1}^{n} d_i X_{U(i)}$$

and the corresponding variances and covariances are

$$Var(\hat{\mu})=\sigma^2 \frac{\alpha_n b_n}{D} \tag{2.15}$$

$$Var(\hat{\sigma})=\sigma^2 \frac{b_n^2}{D} T \tag{2.16}$$

$$Cov(\hat{\mu},\hat{\sigma})=-\sigma^2 \frac{b_n}{D} \tag{2.17}$$

where

$$c_1=\frac{3}{2}\frac{\alpha_n b_n}{D}, \quad c_i=\frac{1}{2i}\frac{\alpha_n b_n}{D}, \quad i=2,3,\ldots,n-1,$$

$$c_n=1\frac{\alpha_n b_n}{2D}\left[3+\sum_{i=2}^{n-1}\frac{1}{i}\right],$$

$$d_1=\frac{3b_n}{2D}$$

$$d_i=\frac{1}{2i}\frac{b_n}{D}, \quad i=2,3,\ldots,n-1,$$

$$d_n=\frac{b_n}{2D}\left[3+\sum_{i=2}^{n-1}\frac{1}{i}\right],$$

$$D=\alpha_n b_n T-1, \qquad T=\frac{3}{2}+\sum_{i=2}^{n-1}\frac{1}{2i}+(2n-1)\left[\frac{b_{n-1}}{b_n}-1\right]$$

$$\alpha_n=\sqrt{2}\,\frac{\Gamma(n+1/2)}{\Gamma(n)}, \qquad b_n=\sqrt{2}\,\frac{\Gamma(n+1)\,\Gamma(n+1/2)}{\Gamma(n+1/2)\,\Gamma(n)}.$$

2.3: EXPONENTIAL DISTRIBUTION (*Ahsanullah, 2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from an exponential distribution with *pdf*

$$f(x,\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \text{ , } x \geq \mu, \sigma > 0.$$

Let $Y_i = \sigma^{-1}(X_{U(i)} - \mu), i = 1,2,\ldots,n,$ (2.18)

then

$$E(Y_i) = i = Var(Y_i), i = 1,2,\ldots,n, \text{ and } Cov(Y_i, Y_j) = \min(i, j).$$

Let $X' = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$ then

$$E(X) = \mu 1 + \sigma\alpha \tag{2.19}$$

$$Var(X) = \sigma^2 \Sigma, \ \alpha = (1,2,\ldots,n)$$

The inverse of variance-covariance $\Sigma^{-1}(=\sigma^{ij})$ can be expressed as

$$\sigma^{ij} = \begin{cases} 2 & if\ i = j = 1,2,\ldots,n-1 \\ 1 & if\ i = j = n \\ -1 & if\ |i-j| = 1, i,j = 1,2,\ldots,n \\ 0 & otherwise. \end{cases} \tag{2.20}$$

It can be shown that for exponential distribution

$$1'\Sigma^{-1} = (1,0,0,\ldots,0), \ \alpha'\Sigma^{-1} = (0,0,0,\ldots,1),$$

$$\alpha'\Sigma^{-1}\alpha = n \qquad \text{and} \qquad \Delta = n-1.$$

On simplification, we get the BLUEs $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively

$$\hat{\mu} = (nX_{U(1)} - X_{U(n)})/(n-1) \tag{2.21}$$

$$\hat{\sigma} = (X_{U(n)} - X_{U(1)})/(n-1) \tag{2.22}$$

with

$$Var(\hat{\mu}) = n\sigma^2 /(n-1), \quad Var(\hat{\sigma}) = \sigma^2 /(n-1)$$

and

$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2 /(n-1).$$

2.4: GENERALIZED PARETO DISTRIBUTION (*Ahsanullah, 2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from $GPD\,(\mu,\sigma,\beta)$ with *pdf*

$$f(x,\mu,\sigma,\beta) = \frac{1}{\sigma}\left\{1+\beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-(1+1/\beta)} \qquad ,x \ge \mu, for\ \beta > 0.$$

Let $X' = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$, then

$$E(X) = \mu\,1 + \sigma\alpha$$

$$Var(X) = \sigma^2\Sigma, \quad \alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$\alpha_i = \beta^{-1}(1-\beta)^{-i}, \qquad i = 1,2,\ldots,n \tag{2.23}$$

$$Var(X) = \sigma^2\Sigma, \Sigma = (\sigma_{i\,j}), \sigma_{i\,j} = \beta^{-2}a_i b_j, 1 \le i \le j \le n$$

and $\sigma_{i,j} = \sigma_{j,i}$.

We can express inverse of Σ as $\Sigma^{-1}(=\sigma^{i\,j})$

$$\sigma^{i+1,i} = \sigma^{i,i+1} = \frac{-1}{a_{i+1}b_i - a_i\,b_{i+1}} = -(1-2\beta)^{i+1}(1-\beta),$$

$$i = 1,2,\ldots,n-1 \tag{2.24}$$

$$\sigma^{i,i} = \frac{a_{i+1}\,b_{i-1} - a_{i-1}\,b_{i+1}}{(a_i\,b_{i-1} - a_{i-1}\,b_i)(a_{i+1}\,b_i - a_i\,b_{i+1})} \quad , \ i = 1,2,\ldots,n \tag{2.25}$$

$$\sigma^{i,j} = 0 \ \text{ for } \ |i-j| > 1.$$

On simplification, we obtain

$$\sigma^{i,i} = (1-2\beta)^i(2-4\beta+\beta^2), \qquad i=1,2,\ldots,n-1 \tag{2.26}$$

and

$$\sigma^{n,n} = (1-2\beta)^n(1-\beta). \tag{2.27}$$

On substituting the values for α and Σ^{-1} in (2.1) and (2.2) and after simplification, it can be shown that

$$\hat{\mu} = X_{U(1)} - \hat{\sigma}(1-\beta)^{-1} \tag{2.28}$$

$$\hat{\sigma} = (1-\beta)(\beta - D^{-1}(1-2\beta)^3 X_{U(1)}) + D^{-1}\beta(1-\beta)\sum_{i=2}^{n}(1-2\beta)^{i+1} X_{U(i)} \tag{2.29}$$

where $D = \sum_{i=2}^{n}(1-2\beta)^{i+1}$.

The corresponding variance and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{T}{D} \tag{2.30}$$

$$Var(\hat{\sigma}) = \sigma^2 \frac{\beta T - (1-2\beta)}{D} \tag{2.31}$$

$$Cov(\hat{\mu},\hat{\sigma}) = \sigma^2 \frac{\{(1-2\beta)^2 + \beta^2 T\}}{D} \tag{2.32}$$

where

$$T = \sum_{i=2}^{n}(1-2\beta)^i.$$

2.5: POWER FUNCTION DISTRIBUTION (*Ahsanullah, 2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from a power function with *pdf*

$$f(x) = \gamma\sigma^{-\gamma}(\mu+\sigma-x)^{\gamma-1}, \qquad \mu < x < \mu+\sigma,\ \sigma>0, \gamma>0.$$

Let $W_i = c_i(X_{U(i)} - \frac{\gamma}{\gamma+1} X_{U(i-1)}), \qquad i=1,2,\ldots,n \qquad (2.33)$

with $X_{U(0)}=0$ and $c_i=(\gamma+1)\left(\frac{\gamma+2}{\gamma}\right)^{\frac{k}{2}}, i=1,2,\ldots,n.$

Then $E(W_1)=\left(\frac{\gamma+2}{\gamma}\right)^{1/2}\{(\gamma+1)\mu+\sigma\},$ (2.34)

$$E(W_i)=\left(\frac{\gamma+2}{\gamma}\right)^{i/2}(\mu+\sigma), i=2,3,\ldots,n \tag{2.35}$$

$$Var(W_i)=\sigma^2, i=1,2,\ldots,n \tag{2.36}$$

$$Cov(W_i,W_j)=0, i\neq j, 1\leq i, j\leq m.$$

Let $W'=(W_1,W_2,\ldots,W_n)$, then $E(W)=X\theta$ where

$$X=\begin{bmatrix}((\gamma+2/\gamma)^{1/2})(\gamma+1) & (\gamma+2/\gamma)^{1/2}\\ (\gamma+2)/\gamma & (\gamma+2)/\gamma\\ \vdots & \vdots\\ (\gamma+2/\gamma)^{n/2} & (\gamma+2/\gamma)^{n/2}\end{bmatrix}, \quad \theta=\begin{bmatrix}\mu\\ \sigma\end{bmatrix}$$

We can write $X'X$ as

$$X'X=\begin{pmatrix}(\gamma+2)^2+T & \gamma+2+T\\ \gamma+2+T & T\end{pmatrix}$$

where

$$T=\sum_{i=1}^{n}(\gamma+2/\gamma)^i$$

therefore,

$$(X'X)^{-1}=D_0^{-1}\begin{pmatrix}T & -(\gamma+2+T)\\ -(\gamma+2+T) & (\gamma+2)^2+T\end{pmatrix}$$

$$D_0=(\gamma+2)(\gamma T-\gamma-2)$$

and

$$X'W = \begin{pmatrix} (\gamma(\gamma+2))^{1/2}W_1 + \sum_{i=1}^{n}\left(\frac{\gamma+2}{\gamma}\right)^{i/2} W_i \\ \sum_{i=1}^{n}\left(\frac{\gamma+2}{\gamma}\right)^{i/2} W_i \end{pmatrix}$$

The BLUEs $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively based on $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ (assuming γ as known) are

$$\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix} = (X'X)^{-1} X'W.$$

On simplification, we get

$$\hat{\mu} = \frac{1}{D_0}\left[(\gamma(\gamma+2))^{1/2}W_1 - \sum_{i=1}^{n}\left(\frac{\gamma+2}{\gamma}\right)^{i/2} W_i \right]. \tag{2.37}$$

$$\hat{\sigma} = \frac{1}{D_0}\left[-(T+\gamma+2)(\gamma(\gamma+2))^{1/2}W_1 + (\gamma+2)(\gamma+1)\sum_{i=1}^{n}\left(\frac{\gamma+2}{\gamma}\right)^{i/2} W_i \right]. \tag{2.38}$$

The variance and covariance of $\hat{\mu}$ and $\hat{\sigma}$ are given by

$$Var(\hat{\mu}) = \sigma^2 T D_0^{-1}, \tag{2.39}$$

$$Var(\hat{\sigma}) = \sigma^2((\gamma+2)^2 T) D_0^{-1} \tag{2.40}$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2(\gamma+2+T) D_0^{-1}. \tag{2.41}$$

BEST LINEAR UNBIASED PREDICTION FOR ORDER STATISTICS AND RECORD VALUES

1. INTRODUCTION

In this chapter, BLUPs of scale and location parameters for exponential, generalized Pareto, power function, Weibull and Rayleigh distribution have been considered for order statistics and record values.

2. BEST LINEAR UNBIASED PREDICTION FOR ORDER STATISTICS

Kaminsky and Nelson (1975) have defined best linear unbiased predictor (BLUP) of $X_{s:n}$ based on $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$, $1 \le r < s \le n$, as $\hat{X}_{s:n}$ if and only if

1. $E(X_{s:n} - \hat{X}_{s:n}) = 0$
2. $E(X_{s:n} - \hat{X}_{s:n})^2$ is minimum

for a continiuous population with *pdf* $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$, and have shown that

$$\hat{X}_{s:n} = \hat{E}(X_{s:n}) + W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha), \tag{2.1}$$

where $1' = (1, 1, \ldots, 1)$,

$$X' = (X_{1:n}, X_{2:n}, \ldots, X_{r:n}),$$

$$\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_r)$$

$$Z_{i:n} = \frac{X_{i:n} - \mu}{\sigma}, \quad i = 1, 2, \ldots, n$$

$$E(Z_{i:n}) = \alpha_i, \qquad i = 1, 2, \ldots, n$$

$$W_i = Cov(Z_{i:n}, Z_{s:n}), \qquad i = 1, 2, \ldots, r \ , \ W' = (W_1, W_2, \ldots, W_r)$$

$$\Sigma = (\sigma_{ij}), \quad \sigma_{ij} = Cov(X_{i:n}, X_{j:n}), \quad i, j = 1, 2, \ldots, r$$

$$Var - Cov(X) = \sigma^2 \Sigma.$$

and $\hat{E}(X_{s:n}) = \hat{\mu} + \hat{\sigma}\alpha_s$ is the BLUE of $E(X_{s:n})$. (2.2)

Clearly $\hat{E}(X_{s:n})$ provides an unbiased predictor of $X_{s:n}$ but, of course, its mean squared error (*mse*) will exceed to that $\hat{X}_{s:n}$. It is well known that the best (unrestricted) least-square predictor of $X_{s:n}$ is

$$\hat{X}_{s:n} = E(X_{s:n} \mid X_{1:n}, X_{2:n}, \ldots, X_{r:n}), \tag{2.3}$$

but $\hat{X}_{s:n}$ in general depends on the unknown parameters. However, its *mse* does provide a lower bound for the error in predicting $X_{s:n}$.

It can be seen that

$$W'\Sigma^{-1} = (0, 0, \ldots, q),$$

where $q = \dfrac{W_i}{\sigma_{ir}}, \quad i = 1, 2, \ldots, r$

for exponential, Pareto and power function distributions and therefore for these distributions

$$\hat{X}_{s:n} = qX_{r:n} + (1-q)\hat{\mu} + (\alpha_s - q\alpha_r)\hat{\sigma}. \tag{2.4}$$

THEOREM 2.1: (*Khan and Abouammoh, 2000*)

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$. Then for *df* $\bar{F}(x) = [ax+b]^c, \alpha < x < \beta$

$$E(X_{s:n} \mid X_{r:n} = x) = a^* x + b^* \tag{2.5}$$

where $a^* = \displaystyle\prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c(n-r-j)+1}, \quad b^* = -\frac{b}{a}(1-a^*).$

PROOF: The conditional *pdf* of $X_{s:n}$ given $X_{r:n} = x$, ($r < s$) is

$$f(X_{s:n} \mid X_{r:n} = x)$$

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s)!} \left(1 - \frac{\bar{F}(y)}{\bar{F}(x)}\right)^{s-r-1} \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{n-s} \frac{f(y)}{[\bar{F}(x)]},$$

$$x \le y \qquad (2.6)$$

and

$$E(X_{s:n} \mid X_{r:n} = x)$$

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s)!} \int_x^\beta y \left(1 - \frac{\bar{F}(y)}{\bar{F}(x)}\right)^{s-r-1} \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{n-s} \frac{f(y)}{[\bar{F}(x)]} dy. \qquad (2.7)$$

Now for $\bar{F}(x) = [ax+b]^c$, (2.7) can be written as

$$E(X_{s:n} \mid X_{r:n} = x)$$

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s)!} \int_x^\beta y \left[1 - \left(\frac{ay+b}{ax+b}\right)^c\right]^{s-r-1} \left(\left(\frac{ay+b}{ax+b}\right)\right)^{c(n-s)} \frac{ac(ay+b)^{c-1}}{(ax+b)} dy. \qquad (2.8)$$

Let $t = \left(\frac{ay+b}{ax+b}\right)^c$, then R.H.S. is

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s)!} \int_0^1 \frac{t^{1/c}(ax+b)-b}{a} (1-t)^{s-r-1}\, t^{n-s}\, dt$$

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s)!} \left[\int_0^1 t^{n-s+1/c} (1-t)^{s-r-1}\, dt - \frac{b}{a} \int_0^1 t^{n-s} (1-t)^{s-r-1}\, dt + \frac{b}{a} \int_0^1 t^{n-s+1/c} (1-t)^{s-r-1}\, dt \right] \qquad (2.9)$$

which reduces to

$$E(X_{s:n} \mid X_{r:n} = x) = a^* x + b^* \qquad (2.10)$$

where $a^* = \prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c(n-r-j)+1}$, $b^* = -\frac{b}{a}(1-a^*)$.

Hence proved.

Since order statistics possess the property of Markovian process, therefore

$$E[X_{s:n} \mid X_{1:n}, X_{2:n}, \ldots, X_{r:n} = x] = E[X_{s:n} \mid X_{r:n} = x].$$

Thus (2.10) may be used to predict $X_{s:n}$ on the basis of $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$, $r < s$.

2.1: EXPONENTIAL DISTRIBUTION (*Ahsanullah and Nevzorov, 2005*)

Suppose that $X_{1:n}, X_{2:n}, \ldots, X_{r:n}, (1 < r < n)$ are the order statistics from an exponential distribution with *df* $F(x)$ given as

$$F(x) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)}, \quad -\infty < \mu < x < \infty, \quad \sigma > 0, \tag{2.11}$$

Then we have

$$\alpha_k = \sigma^{-1} E(X_{k:n} - \mu) = \sum_{j=1}^{k} \frac{1}{n-j+1}, \tag{2.12}$$

and

$$\hat{\mu} = X_{1:n} - \frac{\hat{\sigma}}{n} \tag{2.13}$$

where

$$\hat{\sigma} = \frac{1}{r-1} \left\{ \sum_{i=1}^{r} X_{i:n} - n X_{1:n} + (n-r) X_{r:n} \right\} \tag{2.14}$$

is the BLUP of σ.

Further, $\sigma^2 W_i = Cov(X_{i:n}, X_{s:n}) = \sigma^2 \sum_{k=1}^{i} \frac{1}{(n-k+1)^2}$ (2.15)

$$
\Sigma^{-1} = \begin{pmatrix}
(n-1)^2+n^2 & -(n-1)^2 & \ldots & 0 \\
-(n-1)^2 & (n-2)^2+(n-1)^2 & \ldots & 0 \\
0 & -(n-2)^2 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
. & . & . & . \\
0 & 0 & \ldots & -(n-r+1)^2 \\
0 & 0 & \ldots & (n-r+1)^2
\end{pmatrix} \tag{2.16}
$$

Hence, the BLUP $\hat{X}_{s:n}$ of $X_{s:n}$ is

$$
\begin{aligned}
\hat{X}_{s:n} &= \hat{\mu} + \alpha_s \hat{\sigma} + (X_{r:n} - \hat{\mu} - \alpha_r \hat{\sigma}) \\
&= X_{r:n} + (\alpha_s - \alpha_r)\hat{\sigma} \\
&= X_{r:n} + \sigma \sum_{j=r}^{s-1} \frac{1}{(n-j)}.
\end{aligned} \tag{2.17}
$$

This could have been easily obtained using Theorem 2.1, as comparing $F(x) = e^{-(x-\mu)/\sigma}$ with $\overline{F}(x) = [ax+b]^c$, we get

$$
a = -\frac{1}{\sigma c}, \quad b = \frac{\sigma c + \mu}{\sigma c}, \quad c \to \infty
$$

$$
a^* = 1, \qquad b^* = \sigma \sum_{j=r}^{s-1} \frac{1}{(n-j)}. \tag{2.18}
$$

Thus the predicted value of $X_{s:n}$ in case of exponential distribution is given as

$$
\begin{aligned}
E[X_{s:n} \mid X_{r:n} = x] &= a^* x + b^* \\
&= x + \sigma \sum_{j=r}^{s-1} \frac{1}{(n-j)}
\end{aligned}
$$

as obtained in (2.17) when σ is known. If σ is not known then it is to be replaced by its BLUE given in (2.14).

2.2: UNIFORM DISTRIBUTION (*Ahsanullah and Nevzorov, 2005*)

Suppose that $X_{1:n}, X_{2:n}, \ldots, X_{r:n}, (1 < r < n)$ are the order statistics from a uniform distribution with *df* $F(x)$ given as

$$F(x) = \frac{2x - 2\mu + \sigma}{2\sigma}, \quad \mu - \frac{\sigma}{2} < x < \mu + \frac{\sigma}{2},$$

$$\sigma > 0, -\infty < \mu < \infty. \quad (2.19)$$

Let $\quad Y = \dfrac{X - \mu}{\sigma}$, then $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$

then

$$E(Y_{i:n}) = \alpha_i = \frac{i}{n+1} - \frac{1}{2}, i = 1, 2, \ldots, r, s, \quad (2.20)$$

$$\hat{\mu} = \frac{(2r - n - 1)X_{1:n} + (n - 1)X_{r:n}}{2(r-1)} \quad (2.21)$$

and

$$\hat{\sigma} = \frac{n+1}{r-1}(X_{r:n} - X_{1:n}). \quad (2.22)$$

Further,

$$W_i = \frac{i(n - s + 1)}{(n+1)^2(n+2)}, \quad (2.23)$$

$$\Sigma^{-1} = (n+1)(n+2)\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ldots & 0 \\ . & . & . & . & . & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & \dfrac{n-r+2}{n-r+1} \end{pmatrix}$$

$$(2.24)$$

$$W'\Sigma^{-1} = \left(0,0,\ldots,\frac{n-s+1}{n-r+1}\right). \tag{2.25}$$

Hence, the BLUP $\hat{X}_{s:n}$ of $X_{s:n}$ is

$$\hat{X}_{s:n} = \hat{\mu} + \alpha_s\hat{\sigma} + \frac{n-s+1}{n-r+1}(X_{r:n} - \hat{\mu} - \alpha_r\hat{\sigma}). \tag{2.26}$$

This could have been easily obtained using Theorem 2.1, as comparing

$$\bar{F}(x) = \frac{-2x+2\mu+\sigma}{2\sigma},$$

with $\bar{F}(x) = [ax+b]^c$, we get

$$a = -\frac{1}{\sigma}, \quad b = \frac{\mu}{\sigma} + \frac{1}{2}, \quad c = 1$$

then we have

$$a^* = \prod_{j=0}^{s-r-1} \frac{(n-r-j)}{(n-r-j)+1} = \frac{n-s+1}{n-r+1}, \quad b^* = (\mu + \frac{\sigma}{2})(1-a^*). \tag{2.27}$$

Hence the predicted value of $X_{s:n}$ in this case, for known μ and σ, is given by

$$E[X_{s:n} \mid X_{r:n} = x] = a^* X_{r:n} + b^*.$$

For unknown μ and σ, replace μ and σ by their BLUEs

$$\hat{\mu} = \frac{(2r-n-1)X_{1:n} + (n-1)X_{r:n}}{2(r-1)}$$

$$\hat{\sigma} = \frac{n+1}{r-1}(X_{r:n} - X_{1:n})$$

as given in (2.21) and (2.22).

2.3: PARETO DISTRIBUTION

$$\bar{F}(x)=\left(\frac{\alpha+\delta}{x+\delta}\right)^{\theta},\ \alpha\le x<\infty,\ \theta>0.$$

On comparing it with $\bar{F}(x)=[ax+b]^{c}$, we get

$$a=\frac{1}{\alpha+\delta},\quad b=\frac{\delta}{\alpha+\delta},\quad c=-\theta$$

$$a^*=\prod_{j=0}^{s-r-1}\frac{\theta(n-r-j)}{\theta(n-r-j)-1}>1,\quad b^*=-\delta(1-a^*). \tag{2.28}$$

Predicted value of $X_{s:n}$ in this case, for known θ and δ is given as

$$E[X_{s:n}\mid X_{r:n}=x]=a^*X_{r:n}+b^*.$$

If θ and δ are not known then they are to be replaced by their BLUEs.

2.4: POWER FUNCTION DISTRIBUTION

$$\bar{F}(x)=\left(\frac{\mu+\sigma-x}{\sigma}\right)^{\gamma}=\left[-\frac{1}{\sigma}x+\frac{\mu+\sigma}{\sigma}\right]^{\gamma},\ \mu\le x\le\mu+\sigma.$$

Here

$$a=-\frac{1}{\sigma},b=\frac{\mu+\sigma}{\sigma},c=\gamma$$

and

$$a^*=\prod_{j=0}^{s-r-1}\frac{\gamma(n-r-j)}{\gamma(n-r-j)+1}<1,\ b^*=(\mu+\sigma)(1-a^*). \tag{2.29}$$

Then predicted value of $X_{s:n}$ in this case, for known μ and σ, is given by

$$E[X_{s:n}\mid X_{r:n}=x]=a^*X_{r:n}+b^*.$$

If θ and δ are not known then they are to be replaced by their BLUEs.

If $h(x)$ is a monotonic measurable function of X, then it can be seen that

$$E[X_{s:n} \mid X_{r:n} = x] = a^* X_{r:n} + b^* \text{ for } \bar{F}(x) = [ax+b]^c$$

implies that

$$E[h(X_{s:n}) \mid X_{r:n} = x] = a^* h(x) + b^*$$

for

$$\bar{F}(x) = [ah(x)+b]^c$$

and therefore $X_{s:n}$ can be predicted by inverting for the following distributions as well.

Table 1: Examples based on $\bar{F}(x) = [ah(x)+b]^c$

Distribution	$F(x)$	a	b	c	$h(x)$
Power function	$a^{-p}x^p, 0 \le x \le a$	$-a^{-p}$	1	1	x^p
Pareto	$1-a^p x^{-p}, a \le x \le \infty$	a^{-1}	0	$-p$	x
Beta of first Kind	$1-(1-x)^p, 0 \le x \le 1$	-1	1	p	x
Weibull	$1-e^{-\theta x^p}, 0 \le x < \infty$	1	0	1	$e^{-\theta x^p}$
Inverse Weibull	$e^{-\theta x^{-p}}, 0 \le x < \infty$	-1	1	1	$e^{-\theta x^{-p}}$
Burr type II	$(1+e^{-x})^{-k}, -\infty < x < \infty$	-1	1	1	$(1+e^{-x})^{-k}$
Burr type III	$(1+x^{-c})^{-k}, 0 \le x < \infty$	-1	1	1	$(1+x^{-c})^{-k}$
Burr type IV	$\left[1+\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{-k}$, $0 \le x \le c$	-1	1	1	$\left[1+\left(\frac{c-x}{x}\right)^{-c}\right]^{-k}$

Burr type V	$[1+ce^{-\tan x}]^{-k}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	-1	1	1	$[1+ce^{-\tan x}]^{-k}$
Burr type VI	$[1+ce^{-k\sinh x}]^{-k}$, $-\infty < x < \infty$	-1	1	1	$[1+ce^{-k\sinh x}]^{-k}$
Burr type VII	$2^{-k}[1+\tanh x]^k$, $-\infty < x < \infty$	-2^{-k}	1	1	$[1+\tanh x]^k$
Burr type VIII	$\left(\frac{2}{\pi}\tan^{-1} e^x\right)^k, -\infty < x < \infty$	$-\left(\frac{2}{\pi}\right)^k$	1	1	$(\tan^{-1} e^x)^k$
Burr Type IX	$1-\frac{2}{c[(1+e^x)^k-1]+2}$, $-\infty < x < \infty$	$\frac{c}{2}$	$1-\frac{c}{2}$	-1	$(1+e^x)^k$
Burr type X	$(1-e^{-x^2})^k, 0 \le x < \infty$	-1	1	1	$(1-e^{-x^2})^k$
Burr type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^k$, $0 \le x \le 1$	-1	1	1	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^k$
Burr Type XII	$1-(1+\theta x^p)^{-m}, 0 \le x < \infty$	θ	1	$-m$	x^p
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1} x, -\infty < x < \infty$	$-\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1} x$

EXAMPLE: For power function distribution

$$\hat{X}^p_{s:n} = E[X^p_{s:n} \mid X_{r:n} = x] = a^* x^p + b^*$$

$$a^* = \prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c\,(n-r-j)+1} = \frac{n-s+1}{n-r+1}, b^* = \frac{a^p(s-r)}{n-r+1}$$

and $\hat{X}_{s:n} = [a^* x^p + b^*]^{1/p}$.

3. Best Linear Unbiased Prediction Of Record Values

Let X_n, $n \geq 1$ be a sequence of *i.i.d.* continuous random variables with *pdf* $f(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$. Let $X_{U(s)}$ be the s^{th} upper record value. We will predict the s^{th} upper record value on the basis of first r record values for $s > r$.

Let $W' = (W_1, W_2, \ldots, W_r)$, where

$$\sigma^2 W_i = Cov(X_{U(i)}, X_{U(s)}),\ i = 1, 2, \ldots, r \text{ and}$$

$$\alpha_s = \sigma^{-1} E(X_{U(s)} - \mu) \tag{3.1}$$

The BLUP of $X_{U(s)}$ is (Ahsanullah and Nevzorov, 2005)

$$\hat{X}_{U(s)} = \hat{\mu} + \hat{\sigma}\alpha_s + W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha),$$

where $\hat{\mu}$, $\hat{\sigma}$ are the BLUEs of μ, σ respectively.

Theorem 3.1: (*Athar et al.*, 2003)

Let X be an absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$ on the support(α, β), where α and β may be finite or infinite. Then for $r < s$,

$$E[X_{U(s)} \mid X_{U(r)} = x] = a^* x + b^* \tag{3.2}$$

iff

$$\bar{F}(x) = [ax + b]^c \quad x \in (\alpha, \beta) \tag{3.3}$$

where $a^* = \left(\frac{c}{c+1}\right)^{s-r}$ and $b^* = -\frac{b}{a}[1 - a^*]$.

Proof: We have $\bar{F}(x) = [ax+b]^c, f(x) = -ac[ax+b]^{c-1}$

Now, from (3.1)

$$E[X_{U(s)} \mid X_{U(r)} = x]$$

$$= \frac{1}{\Gamma(s-r)} \int_x^{\beta} y\,[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} \frac{f(y)}{\bar{F}(x)}\, dy$$

$$= \frac{1}{\Gamma(s-r)[ax+b]^c} \int_x^{\beta} y\left[c \ln\left(\frac{ax+b}{ay+b}\right)\right]^{s-r-1} ac[ay+b]^{c-1} dy.$$

Let $t = \ln\left(\frac{ax+b}{ay+b}\right)^c$, then R.H.S. is

$$\frac{1}{a\,\Gamma(s-r)} \int_0^{\infty} [(ax+b)e^{-t/c} - b]\, t^{s-r-1}\, e^{-t}\, dt$$

which reduces to

$$\left(\frac{c}{c+1}\right)^{s-r} x + \frac{b}{a}\left[\left(\frac{c}{c+1}\right)^{s-r} - 1\right] \tag{3.4}$$

hence proved.

This result will be used to predict $X_{U(s)}$ on the basis of $X_{U(1)}, X_{U(2)}, \ldots, X_{U(r)}$, $r < s$, noting that

$$E[X_{U(s)} \mid X_{U(1)}, X_{U(2)}, \ldots, X_{U(r)} = x] = E[X_{U(s)} \mid X_{U(r)} = x]$$

because of the Markovian property of record statistics.

3.1: EXPONENTIAL DISTRIBUTION (*Ahsanullah, 2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from an exponential distribution with *pdf*

$$f(x,\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}\ , \ x \geq \mu, \sigma > 0$$

It can be shown that $W'\Sigma^{-1}(X - \mu 1 - \sigma\alpha) = 0$.

Then $\hat{X}_{U(s)} = ((s-1)X_{U(r)} + (r-s)X_{U(1)})/(r-1)$ (3.5)

$$E(\hat{X}_{U(s)}) = \mu + s\sigma$$

$$Var(\hat{X}_{U(s)}) = \sigma^2(r+s^2-2s)/(r-1)$$

$$MSE(\hat{X}_{U(s)}) = E(\hat{X}_{U(s)} - X_{U(s)})^2$$

$$= \sigma^2(s-r)(s-1)/(r-1). \tag{3.6}$$

BLUP of $X_{U(s)}$ can be easily obtained using Theorem 3.1. On comparing $\bar{F}(x) = e^{-\left(\frac{x-\mu}{\sigma}\right)}$ with $\bar{F}(x) = [ax+b]^c$, we get

$$a = -\frac{1}{\sigma c}, \quad b = \frac{\sigma c + \mu}{\sigma c}, \quad c \to \infty \tag{3.7}$$

$$a^* = 1, \; b^* = \sigma(s-r) \tag{3.8}$$

Thus for exponential distribution the predicted value of $X_{U(s)}$ is

$$E[X_{U(s)} \mid X_{U(r)} = x] = X_{U(r)} + \sigma(s-r) \tag{3.9}$$

when σ is known. If σ is not known it is to be replaced by its BLUE

$$\hat{\sigma} = \frac{X_{U(r)} - X_{U(1)}}{r-1}$$

to get expression as given in (3.5).

3.2: POWER FUNCTION DISTRIBUTION (*Ahsanullah*, *2004*)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from a power function with *pdf*

$$f(x) = \gamma\sigma^{-\gamma}(\mu+\sigma-x)^{\gamma-1}, \qquad \mu < x < \mu+\sigma, \sigma > 0, \gamma > 0$$

We will consider the predictor of $X_{U(s)}$ based on r record values for $r < s$. Let $X = (X_{U(1)}, X_{U(2)}, \ldots, X_{U(r)})$, then we can write

$$E(X) = \mu 1 + \sigma\alpha \tag{3.10}$$

where $1' = (1,1,\ldots,1), \alpha' = (\alpha_1,\alpha_2,\ldots,\alpha_r)$ and $\alpha_i = \left(\frac{\gamma}{\gamma+1}\right)^i - 1$.

$$Var(X) = \sigma^2 \Sigma, \quad \Sigma = (\sigma_{i,j}), \quad \sigma_{i,j} = a_i b_j$$

$$a_i = \left(\frac{\gamma+1}{\gamma}\right)^i \left[\left(\frac{\gamma}{\gamma+2}\right)^i \left(\frac{\gamma}{\gamma+1}\right)^{2i}\right] \text{ and } b_j = \left(\frac{\gamma+1}{\gamma}\right)^j.$$

Let $\Sigma^{-1} = (\sigma^{i,j})$, then it can be shown that

$$(\sigma^{i,j}) = \begin{cases} -\left(\frac{\gamma+2}{\gamma}\right)^{\min(i,j)} (\gamma+2)(\gamma+1), |i-j|=1, i,j=1,2,\ldots,r \\ \left(\frac{\gamma+2}{\gamma}\right)^{i+1} (2\gamma^2+4\gamma+1)\frac{\gamma(\gamma+2)}{(\gamma+1)^2}, i=1,2,\ldots,r-1 \\ \left(\frac{\gamma+2}{\gamma}\right)^{n-1} (\gamma+2)(\gamma+1)\left(\frac{\gamma+1}{\gamma}\right), i=n \\ 0, |i-j|>1 \end{cases} \tag{3.11}$$

Let $W' = (W_1, W_2, \ldots, W_r)$, where

$$\sigma^2 W_i = Cov(X_{U(i)}, X_{U(s)}), i = 1,2,\ldots,r \text{ and}$$

$$\alpha_s = \sigma^{-1} E(X_{U(s)} - \mu) \tag{3.12}$$

Then the best (in sense of minimum mean square error) linear unbiased predictor (BLUP) of $X_{U(s)}$ based on the record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(r)}$ is

$$\hat{X}_{U(s)} = \hat{\mu} + \hat{\sigma}\alpha_s + W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha),$$

$\hat{\mu}$, $\hat{\sigma}$ are the BLUEs of μ, σ respectively. It can be shown that

$$W'\Sigma^{-1} = \left(0, 0, \ldots, \left(\frac{\gamma}{\gamma+1}\right)^{s-r}\right) \tag{3.13}$$

$$W'\Sigma^{-1}(X-\hat{\mu}1-\hat{\sigma}\alpha)=\left(\frac{\gamma}{\gamma+1}\right)^{s-r}(X_{U(r)}-\hat{\mu}-\hat{\sigma}\alpha_r). \tag{3.14}$$

Substituting these values in the expression of $\hat{X}_{U(s)}$, we get

$$\hat{X}_{U(s)}=\left(\frac{\gamma}{\gamma+1}\right)^{s-r}(X_{U(r)}-\hat{\mu}+\hat{\sigma}\alpha_r)+\hat{\mu}-\hat{\sigma}\alpha_s. \tag{3.15}$$

We can easily obtain (3.15) using Theorem 3.1, on comparing $\bar{F}(x)=\left[-\frac{1}{\sigma}x+\frac{\mu+\sigma}{\sigma}\right]^{\gamma}$ with $\bar{F}(x)=[ax+b]^c$, we get

$$a=-\frac{1}{\sigma}, b=\frac{\mu+\sigma}{\sigma}, c=\gamma$$

$$a^*=\left(\frac{\gamma}{\gamma+1}\right)^{s-r}<1,\quad b^*=(\mu+\sigma)\left[1-\left(\frac{\gamma}{\gamma+1}\right)^{s-r}\right] \tag{3.16}$$

Thus the predicted value of $X_{U(s)}$ in this case is

$$E[X_{U(s)}\mid X_{U(r)}=x]=a^*X_{U(r)}+b^* \tag{3.17}$$

if μ and σ are known.

If we replace μ and σ in (3.17) by the BLUEs $\hat{\mu}$ and $\hat{\sigma}$ respectively, we get BLUP of $X_{U(s)}$ as given in (3.15).

3.2: GENERALIZED PARETO DISTRIBUTION (Ahsanullah, 2004)

Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are the n record values from $GPD\ (\mu,\sigma,\beta)$ with *pdf*

$$f(x,\mu,\sigma,\beta)=\frac{1}{\sigma}\left\{1+\beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-(1+1/\beta)} \quad ,x\geq\mu, \text{ for } \beta>0$$

We will predict the s^{th} upper record value based on the first r record values for $s>r$.

Let $W'=(W_1,W_2,\ldots,W_r)$, where

$$\sigma^2 W_i = Cov(X_{U(i)}, X_{U(s)}), i = 1,2,\ldots,r \text{ and}$$

$$\alpha_s = \sigma^{-1} E(X_{U(s)} - \mu) \tag{3.18}$$

The BLUP of $X_{U(s)}$ is $\hat{X}_{U(s)}$, where

$$\hat{X}_{U(s)} = \hat{\mu} + \hat{\sigma}\alpha_s + W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha),$$

$\hat{\mu}$, $\hat{\sigma}$ are the BLUEs of μ, σ respectively. It can be shown that

$$W'\Sigma^{-1}(X - \hat{\mu}1 - \hat{\sigma}\alpha) = (1-\beta)^{n-s}(X_{U(n)} - \hat{\mu} - \hat{\sigma}\alpha_n). \tag{3.19}$$

Thus

$$\hat{X}_{U(s)} = (1 + \beta\alpha_{s-r})X_{U(r)} - \alpha_{s-r}(\beta\hat{\mu} - \hat{\sigma}) \tag{3.20}$$

$$\alpha_{s-r} = \frac{1}{\beta}\{(1-\beta)^{r-s} - 1\}. \tag{3.21}$$

BLUP as given in (3.20) could have also been obtained, using Theorem 3.1, comparing $\bar{F}(x) = \left\{1 + \beta\left(\frac{x-\mu}{\sigma}\right)\right\}^{-1/\beta}$ with $\bar{F}(x) = [ax+b]^c$, to get

$$a = \frac{\beta}{\sigma}, b = \frac{1-\beta\mu}{\sigma}, \ c = -\frac{1}{\beta},$$

and BLUP of $X_{U(s)}$

$$\hat{X}_{U(s)} = E[X_{U(s)} \mid X_{U(r)} = x] = a^* X_{U(r)} + b^*$$

where

$$a^* = \prod_{j=0}^{s-r-1} \frac{(n-r-j)}{(n-r-j)-\beta}, \ b^* = (\mu - \frac{\sigma}{\beta})(1 - a^*).$$

If μ and σ are not known then, they are to be replaced by their BLUEs.

As for order statistics, it can be seen for records as well that, if $h(x)$ is a monotonic measurable function of X, then it can be seen that

$$E[X_{U(s)} \mid X_{U(r)} = x] = a^* X_{U(r)} + b^* \text{ for } \bar{F}(x) = [ax+b]^c$$

implies that

$$E[h(X_{U(s)}) | X_{U(r)} = x] = a^* h(x) + b^*$$

for

$$\bar{F}(x) = [ah(x) + b]^c .$$

Therefore $X_{U(s)}$ may also be predicted for the distributions given in Table 1, as discussed for order statistics.

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